I. Equilibrium of the perfect column  
(column is straight and subject to centric compressive force \( P \))

\[ u_o(x) = -\frac{P}{EA} x \quad v_o(x) = 0 \quad x \in (0, L) \]

II. Stability of the equilibrium state in part (I) by the method of adjacent equilibrium

Consider infinitesimal variation to the displacements represented by subscript “1”  
(e.g., \( w_1(x) = \delta w(x) \))

If the column is fixed at \( x = 0 \)

\[
\begin{align*}
  u(x) &= u_o(x) + u_1(x) \implies u(0) = 0 \implies u_1(x) = 0 \\
  u(x) &= u_o(x) \quad \forall x \in (0, L)
\end{align*}
\]

\[
\begin{align*}
  v(x) &= v_o(x) + v_1(x) \\
  v(x) &= v_1(x) \quad \forall x \in (0, L)
\end{align*}
\]

Equilibrium:

\[
\begin{align*}
  P \\
  M_z \\
  V_y
\end{align*}
\]

\[
\begin{align*}
  \frac{dV_{y_1}}{dx} &= 0 & (1) \\
  \frac{dM_{z_1}}{dx} + V_{y_1} + \frac{d^2 v_1}{dx^2} P &= 0 & (2)
\end{align*}
\]

Differentiate Eq. (2) with respect to \( x \) once:

\[
\begin{align*}
  \frac{d^2 M_{z_1}}{dx^2} + \frac{dV_{y_1}}{dx} + \frac{d^2 v_1}{dx^2} P &= 0 & (3) \\
  \frac{d^2 M_{z_1}}{dx^2} + \frac{d^2 v_1}{dx^2} P &= 0 & (4)
\end{align*}
\]

Hooke’s law:

\[
M_{z_1} = EI_z \frac{d\theta_{z_1}}{dx} = EI_z \frac{d(v'_1)}{dx} \quad (5)
\]
Substituting Eq. (5) into Eq. (4) we get the governing ordinary differential equation for buckling:

\[
\frac{d^2}{dx^2} \left( EI_z \frac{d^2 v_1}{dx^2} \right) + P \frac{d^2 v_1}{dx^2} = 0 \quad v_1 = v_1(x) \quad x \in (0, L)
\]  

(6)

For a column with \( EI_z = \text{constant} \)

\[
EI_z v_1''' + P v_1'' = 0 \Rightarrow v_1''' + \frac{P}{EI_z} \frac{v_1''}{\lambda^2} = 0
\]  

(7)

\[
v_1''' + \lambda^2 v_1'' = 0 \quad v_1 = v_1(x) \quad x \in (0, L)
\]  

(8)

General solution for \( \lambda^2 > 0 \)

\[
v_1(x) = A_1 \sin(\lambda x) + A_2 \cos(\lambda x) + A_3 x + A_4
\]  

(9)

The constants are found by applying the boundary conditions. The following will be needed

\[
v_1'(x) = A_1 \lambda \cos(\lambda x) - A_2 \lambda \sin(\lambda x) + A_3
\]  

(10)

\[
v_1''(x) = -A_1 \lambda^2 \sin(\lambda x) - A_2 \lambda^2 \cos(\lambda x)
\]  

(11)

\[
v_1'''(x) = -A_1 \lambda^3 \cos(\lambda x) + A_2 \lambda^3 \sin(\lambda x)
\]  

(12)

From the hooke’s law Eq. (5)

\[
M_{z_1} = EI_z v_1''(x)
\]  

(13)

\[
= -EI_z [A_1 \lambda^2 \sin(\lambda x) + A_2 \lambda^2 \cos(\lambda x)]
\]  

(14)

From equilibrium Eq. (2)

\[
V_{y_1} = -M_{z_1}' - P v_1' = -EI_z [v_1''' + \lambda^2 v_1']
\]  

(15)

\[
= -EI_z [A_3 \lambda^2]
\]  

(16)

Boundary conditions depend from problem to problem. However, there are three standard boundary condition evaluated at the boundary:

**PINNED:** \( v_1 = 0 \quad \text{and} \quad M_{z_1} = 0 \)

**FREE:** \( M_{z_1} = 0 \quad \text{and} \quad V_{y_1} = 0 \)

**CLAMPED:** \( v_1 = 0 \quad \text{and} \quad v_1' = 0 \)
The uniform column with bending stiffness $EI$ shown below is clamped at $x = 0$ and free at $x = L$.

![Column Diagram]

Determine:
1. Boundary conditions
2. Characteristic equation to determine the buckling loads
3. Determine the critical load $P_{cr}$
4. Determine the buckling mode shape associated with $P_{cr}$

**BOUNDARY CONDITIONS**

Clamped at $x = 0$: $v_1(0) = 0$ and $v'_1(0) = 0$

$$v_1(0) = A_2 + A_4 = 0$$
$$v'_1(0) = A_1 \lambda + A_3 = 0$$

Free at $x = L$: $M_{z_1}(L) = 0 \Rightarrow v''_1(L) = 0$ and $V_{y_1}(L) = 0$

$$v''_1(L) = -A_1 \lambda^2 \sin(\lambda L) - A_2 \lambda^2 \cos(\lambda L) = 0$$
$$V_{y_1}(L) = A_3 \lambda^2 = 0$$
CHARACTERISTIC EQUATION

Write the boundary conditions in a matrix form in terms of the unknown coefficients $A_1, A_2, A_3, A_4$

\[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
\lambda & 0 & 1 & 0 \\
-\lambda^2 \sin(\lambda L) & -\lambda^2 \cos(\lambda L) & 0 & 0 \\
0 & 0 & \lambda^2 & 0
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\] (17)

Non-trivial solution ($\{A\} \neq 0$) requires $\det[C] = 0$. Using co-factor expansions we get the characteristic equation:

\[
\lambda^5 \cos(\lambda L) = 0
\] (18)

CRITICAL LOAD

We proceed to solve the characteristic equation for $\lambda$. The loads is obtained from:

\[
P_m = \lambda_m^2 \frac{EI_z}{z}
\]

\[
\lambda = 0 \quad \Rightarrow \quad P = 0 \quad \text{leads to trivial solution}
\]

\[
\cos(\lambda L) = 0
\]

\[
\lambda L = \frac{(2m-1)\pi}{2}
\]

\[
\lambda_m = \frac{(2m-1)\pi}{2L} \quad \Rightarrow \quad P_m = \left[\frac{(2m-1)\pi}{2L}\right]^2 EI_z
\]

The critical load is

\[
P_{cr} = P_1 = \frac{\pi^2 EI_z}{4L^2}
\]
BUCKLING MODE SHAPES ASSOCIATED WITH $P_{CR}$

Plug-in the value for $\lambda_m$ in Eq. (17) to determine the coefficients. Recall that for our problem $\cos(\lambda_m L) = 0$

$$
\begin{bmatrix}
0 & 1 & 0 & 1 \\
\lambda_m & 0 & 1 & 0 \\
-\lambda_m^2 \sin(\lambda_m L) & 0 & 0 & 0 \\
0 & 0 & \lambda_m^2 & 0
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
$$

(19)

Therefore the coefficient are

row 3 $\rightarrow$ $A_1 = 0$

row 2 $\rightarrow$ $A_3 = 0$

row 1 $\rightarrow$ $A_2 = -A_4$

Now the mode shapes are obtained by substituting these values into

$$
v_1(x) = A_1 \sin(\lambda_m x) + A_2 \cos(\lambda_m x) + A_3 x + A_4
$$

$$
v_1(x) = -A_4 \cos(\lambda_m x) + A_4
$$

$$
v_1(x) = A_4 \left[ 1 - \cos \left( \frac{(2m-1) \pi x}{2L} \right) \right]
$$

Note that $A_4$ remains indeterminate. Therefore the buckling mode shape associated with $P_{cr}$ is

$$
\frac{v_1(x)}{A_4} = 1 - \cos \left( \frac{\pi x}{2L} \right)
$$