STRAIGHT, UNIFORM, UNSWEPT, HIGH ASPECT RATIO, CANTILEVERED WING IN STEADY INCOMPRESSIBLE FLOW

Bisplinghoff, Ashley, and Halfman. Aeroelasticity\textsuperscript{1}, 1983, pp. 427-432.

Let $\alpha(x) = \alpha_0 + \theta(x)$ is the total wing incidence and

$\alpha_0$ – fixed incidence at the wing root. It could be a function of $x$ for a variable in built-in twist, but we will consider it constant along the span.

$\theta(x)$ – twist angle of the wing due to elastic deformation.

Neglect airfoil weight, since we saw for the rigid wing segment that this factor played no role in the divergence dynamic pressure.

From St. Venant torsion theory we have the differential equation of torque as

$$(-T) + \left(T + T_x \frac{dT}{dx} dx \right) + t_x(x) dx = 0$$

$$\frac{dT}{dx} + t_x = 0 \quad (1)$$

where $t_x$ denotes the external torque per unit span. In this case the external torque per unit span is due to the aerodynamic loads acting on the wing.

St. Venant’s torsion theory relates torque to the unit twist as

$$T = G_o J \frac{d\theta_x}{dx} \quad (2)$$

where $G_o J$ is the torsional stiffness of the wing box.

\textsuperscript{1}Almost all definitions and equations presented in this handout are directly taken out of this reference.
The torsion constant for a single-cell, piecewise homogeneous cross section is given by

\[
J = \frac{4 A_{\text{encl}}^2}{\sum_i \int \frac{ds_i}{t_i^*}}
\]

and the modulus-weighted thickness is given by

\[
t_i^* = \frac{G_i t_i}{G_o}
\]

where \(G_o\) is the reference shear modulus.

Substitute Eq. (2) into (1) and use the fact that the wing is uniform along the span to get

\[
G_o J \frac{d^2 \theta}{dx^2} + t_x = 0 \quad \theta = \theta(x) \quad 0 < x < L \quad (3)
\]

**AERODYNAMICS**

Use “strip theory”, which assumes aerodynamic lift and moment at station \(x\) depends only on the angle of attack (or incidence) at \(x\), and is independent of angle of attack at any other spanwise locations. Physically, this is reasonable for high aspect ratio wing. The differential lift and moment acting at the A.C. on a differential element of the wing is shown below.

\[
dL = q \left( c \, dx \right) C_\ell \\
dM_{AC} = q \left( c \, dx \right) C_{mAC}
\]

where \(C_\ell\) is the local lift coefficient and \(C_{mAC}\) is the local moment coefficient about the aerodynamic center.
Hence, the external torque acting on the differential element about the elastic axis is

\[ t_x \, dx = e \, dL + dM_{AC} \]

\[ = \left[ e \, q \, c \, C_\ell + q \, c^2 \, C_{m_{AC}} \right] \, dx \]  

(4)

\[ t_x(x) = e \, q \, c \, C_\ell + q \, c^2 \, C_{m_{AC}} \]  

(5)

According to strip theory

\[ C_\ell = \frac{\partial C_\ell}{\partial \alpha} a_0 \]

\[ C_\ell = a_0 [\alpha_0 + \theta(x)] \]

where \( a_0 \) is the lift curve slope.

Hence, Eq (5) becomes

\[ t_x(x) = e \, q \, a_0 [\alpha_0 + \theta(x)] + q \, c^2 \, C_{m_{AC}} \]  

(6)

Substitute Eq. (6) into (3) and rearrange the terms to get

\[ G_o J \frac{d^2 \theta}{dx^2} + (q \, e \, a_0) \, \theta = -q \, c \, e \, a_0 \, \alpha_0 - q \, c^2 \, C_{m_{AC}} \quad \theta = \theta(x) \quad 0 < x < L \]  

(7)

Equation (7) is the governing, second order, ordinary differential equation (o.d.e.) for \( \theta(x) \) with \( 0 < x < L \). The boundary conditions at \( x = 0 \) and \( x = L \) are to specify either \( \theta \) or \( T \), but not both. For a cantilever wing, which is clamped at the root and free at the tip, the boundary conditions are

\[ \theta(0) = 0 \quad T(L) = G_o J \frac{d\theta}{dx}_{x=L} = 0 \]

\[ \theta(0) = 0 \quad \frac{d\theta}{dx}_{x=L} = 0 \]  

(8)

The general solution of the o.d.e. (7) is the sum of a particular solution and a homogeneous solution.

\[ \theta(x) = \theta_p(x) + \theta_h(x) \]  

(9)
By the method of undetermined coefficients the particular solution is a constant:

\[ \theta_p(x) = -\alpha_0 - \frac{c C_{mAC}}{a_0 e} \] (10)

The homogeneous solution is found by solving:

\[ G_o J \frac{d^2 \theta_h}{dx^2} + \left( q e c a_0 \right) \theta_h = 0 \]

\[ \frac{d^2 \theta_h}{dx^2} + \left( q e c a_0 \right) \frac{G_o J}{\lambda^2} \theta_h = 0 \]

\[ \frac{d^2 \theta_h}{dx^2} + \lambda^2 \theta_h = 0 \] (11)

The solution to Eq. (11) is

\[ \theta_h(x) = A \sin(\lambda x) + B \cos(\lambda x) \] (12)

Therefore the total solution to the o.d.e. (7) is

\[ \theta(x) = A \sin(\lambda x) + B \cos(\lambda x) - \alpha_0 - \frac{c C_{mAC}}{a_0 e} \] (13)

The constants \( A \) and \( B \) are found using the boundary conditions (8)

\[ \theta(0) = 0 = B - \alpha_0 - \frac{c C_{mAC}}{a_0 e} \Rightarrow B = \alpha_0 + \frac{c C_{mAC}}{a_0 e} \] (14)

Taking the derivative of (13)

\[ \frac{d\theta(x)}{dx} = A \lambda \cos(\lambda x) - B \lambda \sin(\lambda x) \]

\[ \theta'(L) = 0 = A \lambda \cos(\lambda L) - B \lambda \sin(\lambda L) \Rightarrow \quad A = B \tan(\lambda L) \]

\[ A = \left[ \alpha_0 + \frac{c C_{mAC}}{a_0 e} \right] \tan(\lambda L) \] (15)
Therefore, the complete solution to the o.d.e. (7) is

\[
\theta(x) = \left( \alpha_0 + \frac{c C_{mAC}}{a_0 e} \right) \left[ \tan(\lambda L) \sin(\lambda x) + \cos(\lambda x) - 1 \right]
\]

\[
\theta(x) = \left( \alpha_0 + \frac{c C_{mAC}}{a_0 e} \right) \left[ \sin(\lambda L) \sin(\lambda x) + \cos(\lambda L) \cos(\lambda x) \right] - 1
\]

\[
\theta(x) = \left( \alpha_0 + \frac{c C_{mAC}}{a_0 e} \right) \left[ \cos(\lambda (L - x)) \cos(\lambda L) - 1 \right]
\]

(16)

Now we calculate the wing’s total angle of attack:

\[
\alpha(x) = \alpha_0 + \theta(x)
\]

\[
\alpha(x) = \alpha_0 + \left( \alpha_0 + \frac{c C_{mAC}}{a_0 e} \right) \left[ \frac{\cos(\lambda (L - x))}{\cos(\lambda L)} - 1 \right]
\]

(17)

The divergence will occur if \( \alpha(x) \rightarrow \infty \). From Eq. (17) we can see that \( \alpha(x) \rightarrow \infty \) when \( \cos(\lambda L) \rightarrow 0 \). Therefore, for divergence:

\[
\cos(\lambda L) = 0
\]

\[
\lambda L = \frac{(2m - 1)\pi}{2} \quad m = 1, 2, ...
\]

The critical value is the lowest root \((m = 1)\):

\[
\lambda_D L = \frac{\pi}{2} = \sqrt{\frac{q_D c e a_0}{G_o J}} L
\]

The wing torsional divergence dynamic pressure is

\[
q_D = \left( \frac{\pi}{2L} \right)^2 \frac{G_o J}{c e a_0}
\]