INEL 3105 Lecture #18

Note Title 11/10/2009

Fasores:

$v(t) = V_M \cos(\omega t + \theta) \rightarrow \hat{V} = V_M \angle\theta$

$= V_m e^{j\theta}$

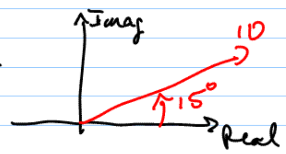
$= V_M \cos\theta + j V_m \sin\theta$

$V_M = \sqrt{V_M^2 \cos^2\theta + V_m^2 \sin^2\theta} = V_M \sqrt{\cos^2\theta + \sin^2\theta} = V_M$

Ejemplo: Evaluar los fasores de:

① $10 \cos(\omega t + 15^\circ)$
② $5 \sin(\omega t + 5^\circ)$
③ $-10 \cos(\omega t + 90^\circ)$; usando como base $\underline{\cos}$:

① $10 \angle 15^\circ$



$10\angle 15^\circ = 10e^{j15} = 10 \cos 15^\circ + j\,10 \sin 15^\circ$

② $5 \sin(\omega t + 5^\circ) = 5 \cos(\underbrace{\omega t + 5^\circ}_{\phi} - 90^\circ)$

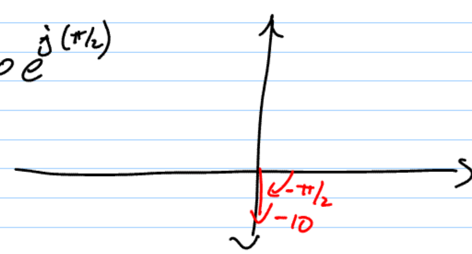
$5 \cos(\phi - 90^\circ) = 5 \cos\phi \cos 90^\circ +$
$5 \sin\phi \sin 90^\circ$
$= 5 \sin\phi = 5 \sin(\omega t + 5)$

$5\angle 5 - 90^\circ = 5 \angle -85^\circ = 5 e^{-j85} = 5 e^{-j\frac{85}{180}\pi}$

③ $-10 \cos(\omega t + 90^\circ) \rightarrow -10 \angle 90^\circ = -10 e^{j(\pi/2)}$

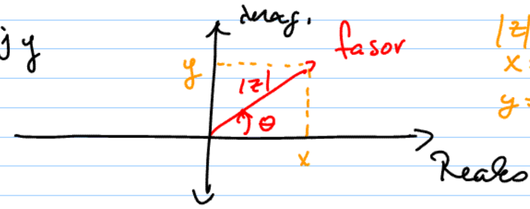
$-10 = 10 \angle 180^\circ = 10 e^{j\pi}$

$-10 e^{j\pi/2} = 10 e^{j\pi} e^{j\pi/2} = 10 e^{j(3\pi/2)}$



Números complejos: parte real & parte imaginaria

$j = \sqrt{-1}$ $\quad z = x + jy$



$|z| = \sqrt{x^2+y^2}$
$x = |z| \cos\theta$
$y = |z| \sin\theta$

$z = |z|\cos\theta + j\,|z|\sin\theta$
$= |z|\,\{\cos\theta + j\sin\theta\}$

EULER : $e^{j\theta} = \cos\theta + j\sin\theta$

$z = |z|e^{j\theta}$ ← forma polar

$|z| = \sqrt{x^2+y^2}$ $\quad \tan\theta = \frac{y}{x}$

$z = x + jy$ ← forma rectangular

Álgebra de #'s complejos:

① Suma: $z = z_1 \pm z_2$; $z_1 = x_1 + jy_1$, $z_2 = x_2 + jy_2$

$z = (x_1 \pm x_2) + j(y_1 \pm y_2) = Z e^{j\theta}$

$|z| = \sqrt{(x_1 \pm x_2)^2 + (y_1 \pm y_2)^2}$; $\theta = \tan^{-1} \frac{y_1 \pm y_2}{x_1 \pm x_2}$

② Multiplicación: $z = z_1 z_2$; $z_1 = |z_1| e^{j\theta_1}$, $z_2 = |z_2| e^{j\theta_2}$ $\quad |z_1| = \sqrt{x_1^2 + y_1^2}$; $\theta_1 = \tan^{-1}\left(\frac{y_1}{x_1}\right)$; $|z_2| = \sqrt{x_2^2+y_2^2}$; $\theta_2 = \tan^{-1}\left(\frac{y_2}{x_2}\right)$

$z = z_1 z_2 = |z_1| e^{j\theta_1} |z_2| e^{j\theta_2}$
$= |z_1||z_2| e^{j(\theta_1+\theta_2)} = Z e^{j\theta}$ → $\theta = \theta_1 + \theta_2$

$|z| = |z_1||z_2|$

(3) División: $z = \frac{z_1}{z_2} = \frac{|z_1| e^{j\theta_1}}{|z_2| e^{j\theta_2}} = \frac{|z_1|}{|z_2|} e^{j\theta_1} - e^{j\theta_2}$

$$z = \frac{|z_1|}{|z_2|} e^{j(\theta_1-\theta_2)} = z e^{j\theta} \rightarrow \theta = \theta_1 - \theta_2$$

$$|z| = \frac{z_1}{z_2}$$

Ejemplo: $z = z_1 + \frac{z_2}{z_3} z_4$

$$z_1 = 4e^{j45°}$$
$$z_2 = 2e^{j90°}$$
$$z_3 = 8e^{j53°}$$
$$z_4 = 4 + j3$$

Solución:

$$\frac{z_2}{z_3} = \frac{2e^{j90°}}{8e^{j53°}} = \frac{1}{4} e^{j(90-53)°} = \frac{1}{4} e^{j37°}$$

$$\left(\frac{z_2}{z_3}\right) z_4 = \frac{1}{4} e^{j37} (4+j3) \quad ; \quad z_4 = \sqrt{4^2+3^2}\, e^{j\tan^{-1}\left(\frac{3}{4}\right)} = 5e^{j37°}$$

$$\frac{z_2}{z_3} z_4 = \frac{1}{4} e^{j37}\, 5e^{j37} = \frac{5}{4} e^{j74°} = \frac{5}{4}\left[\cos 74° + j \sin 74°\right]$$

$$= .344 + j1.20 \quad \text{(en forma rectangular)}$$

$$z_1 = 4\left[4 \cos 45° + j \sin 45°\right] = 4\left[\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right] = 2\sqrt{2} + j2\sqrt{2}$$

$$z = \underbrace{2\sqrt{2} + .344}_{3.17} + j\underbrace{(\sqrt{2} + 1.20)}_{4.028} = 3.17 + j4.028 \quad ; \quad |z| = \sqrt{3.17^2 + 4.028^2} = 5.13$$

$$\theta = \tan^{-1}\left(\frac{4.028}{3.17}\right) = 51.8°$$

$$z = 5.13e^{j51.79°} \leftarrow \text{en forma polar}$$

Fasores:

$$v(t) = V_m \cos(\omega t + \theta)$$
$$= \mathrm{Re}\left\{V_m e^{j(\omega t + \theta)}\right\}$$

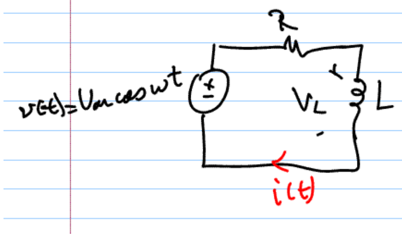
operador de parte real.

$$= Re\{\underbrace{V_m \cos(\omega t+\theta)}_{\text{real}} + j\underbrace{V_m \sin(\omega t+\theta)}_{\text{imaginario}}\} = V_m \cos(\omega t+\theta)$$

$$V_m e^{j(\omega t+\theta)} = \underbrace{V_m e^{j\theta}}_{\text{fasor de } \mho} e^{j\omega t}$$

$$\text{fasor de } \mho = \hat{V} = V_m e^{j\theta}$$

Ejemplo: Evaluar $i(t)$ en régimen permanente



Solución:

$$-v(t) + R i(t) = -L \frac{d i(t)}{dt}$$

$$v(t) = V_m e^{j\omega t}$$
$$i(t) = I_m e^{j(\omega t+\phi)}$$

$$L \frac{d}{dt}\left[I_m e^{j(\omega t+\theta)}\right] + R I_m e^{j(\omega t+\phi)} = V_m e^{j\omega t}$$

$$L I_m j\omega e^{j(\omega t+\phi)} + R I_m e^{j(\omega t+\phi)} = V_m e^{j\omega t}$$

$$j\omega L I_m e^{j\omega t} e^{j\phi} + R I_m e^{j\omega t} e^{j\phi} = V_m e^{j\omega t}$$

$$j\omega L I_m e^{j\phi} + R I_m e^{j\phi} = V_m$$

fasor de $i(t) = \hat{I}$

$$j\omega L \hat{I} + R\hat{I} = V_m$$

$$\hat{I}\{R + j\omega L\} = V_m \quad ; \quad \hat{I} = \frac{V_m}{R + j\omega L} = \frac{V_m e^{j0°}}{\sqrt{R^2 + L^2\omega^2}\; e^{j\tan^{-1}\left(\frac{\omega L}{R}\right)}}$$

$$= \frac{V_m}{\sqrt{R^2 + L^2\omega^2}} e^{j\left(0° - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)}$$

$$\hat{I} = \frac{V_m}{\sqrt{R^2 + L^2\omega^2}} e^{-j\tan^{-1}\left(\frac{\omega L}{R}\right)} = I_m e^{j\phi}$$

$$\therefore \quad I_m = \frac{V_m}{\sqrt{R^2+L^2\omega^2}} \qquad \phi = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$V_m e^{j\omega t} \rightarrow i(t) = I_m e^{j(\omega t+\phi)}$$

$$V_m \cos \omega t \rightarrow i(t) = I_m \cos(\omega t + \phi)$$

$$= \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} \cos\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

excitación: $v(t) = V_m e^{j(\omega t+\theta_v)} \leftarrow Re[v(t)]$

respuesta: $x(t) = X_m e^{j(\omega t+\theta_x)} \leftarrow Re[x(t)]$

excitación: $v(t) = V_m \cos(\omega t + \theta_v) \leftarrow$

respuesta: $x_c(t) = X_m \cos(\omega t + \theta_x) \leftarrow$

ángulo de fase

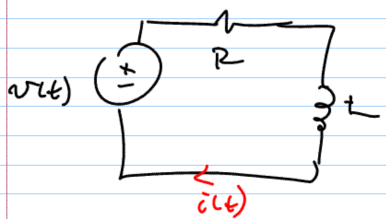
$$\text{Fasor de } v(t) = V_m e^{j\theta_v} = \hat{V} = V_m e^{j(\omega t+\theta_v)}$$

amplitud

$\downarrow$

produce el fasor $\hat{X} = X_m e^{j\theta_x}$; $x(t) = X_m e^{j(\omega t+\theta_x)}$

ejemplo: Eval $i(t)$ en régimen permanente



$$v(t) = V_m \cos \omega t = V_m e^{j\omega t}$$

$$v(t) = Ri + L\frac{di(t)}{dt}$$

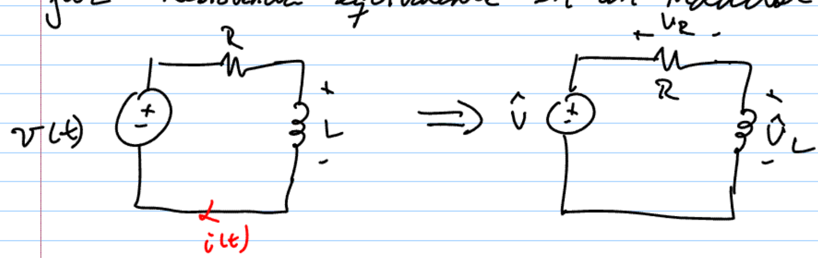
$i(t) = I_m e^{j(\omega t+\theta)}$; Fasor de $i(t) = \hat{I} = I_m e^{j\theta}$

$$V_m \cancel{e^{j\omega t}} = R I_m \cancel{e^{j\omega t}} e^{j\theta} + jL\omega \cancel{e^{j\omega t}} e^{j\theta} I_m$$

$$V_m = R \underbrace{I_m e^{j\theta}} + j\omega L \underbrace{I_m e^{j\theta}}$$

$$\hat{V} = R\hat{I} + \underbrace{j\omega L}_{\text{unidades de ohmios}}\hat{I}$$

$j\omega L$ = Resistencia equivalente en un inductor ; Impedancia = Z_L



$v_R(t) = R\,i(t)$

$\hat{V}_R = R\hat{I}$

$v_L(t) = L\underbrace{\frac{di(t)}{dt}}_{\frac{d}{dt} = j\omega} \Rightarrow \hat{V}_L = j\omega L\hat{I}$

Suma de voltajes fasorial "KVL"

$$\hat{V} - \hat{V}_R - \hat{V}_L = 0 \quad ; \quad \hat{V} = R\hat{I} + j\omega L\hat{I}$$

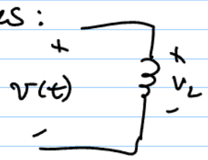
$$\hat{I} = \frac{\hat{V}}{R + j\omega L} \quad ; \quad \hat{I} = \frac{\hat{V}}{\hat{Z}} \quad ; \quad \hat{Z} = R + Z_L$$

$$\hat{Z} = R + j\omega L = |Z|e^{j\theta_z} \qquad |Z| = \sqrt{R^2 + \omega^2 L^2}$$

$$\theta_z = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$\hat{I} = \frac{\hat{V}}{|Z|e^{j\theta_z}} = \frac{V_m}{|Z|}e^{-j\theta_z} \quad ; \quad \hat{I} = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}e^{-j\tan^{-1}\left(\frac{\omega L}{R}\right)} = I_m e^{j\theta}$$

Inductores:



$$\oint E \cdot dl = 0 \; ; \; -v(t) + 0 = -L\frac{di(t)}{dt} \Rightarrow v(t) = L\frac{di(t)}{dt}$$

$$V_m e^{j\omega t}e^{j\theta_v} = L\frac{d}{dt}\left[I_m e^{j\omega t}e^{j\theta_i}\right]$$

$$= L I_m e^{j\theta_i}\, j\omega e^{j\omega t}$$

$$V_m e^{j\theta_v} = j\omega L\, I_m e^{j\theta_i}$$

$$\hat{V} = j\omega L\,\hat{I}_L \quad ; \quad \hat{I}_L = I_m e^{j\theta_i}$$

$\hat{V}_L = j\omega L\,\hat{I}_L$ → impedancia inductiva $Z_L = j\omega L$

$$\boxed{V_L = Z_L \hat{I}_L}$$

$$j = |j| e^{j\measuredangle j} \quad ; \quad |j| = 1$$

$$\measuredangle j = \tan^{-1}\left(\frac{1}{0}\right) = 90^\circ = \pi/2$$

$$\boxed{j = 1e^{j\pi/2}}$$

$$V_m e^{j\theta_v} = j\omega L I_m e^{j\theta_i}$$

$$V_m e^{j\theta_v} = \omega L I_m e^{j\theta_i} 1e^{j\pi/2}$$

$$= \omega L I_m e^{j(\theta_i + \pi/2)}$$

$$V_m = \omega L I_m$$

$$\theta_v = \theta_i + \pi/2$$

v_L "leads" I_m by $\pi/2$