

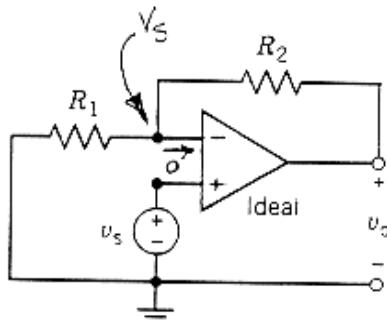
Chapter 6: The Operational Amplifier

Exercises

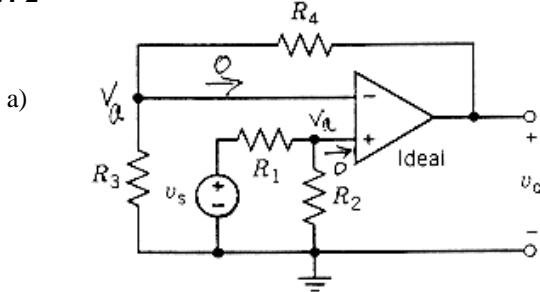
Ex. 6.4-1

$$\frac{v_s}{R_1} + \frac{v_s - v_0}{R_2} + 0 = 0$$

$$\frac{v_0}{v_s} = 1 + \frac{R_2}{R_1}$$



Ex. 6.4-2



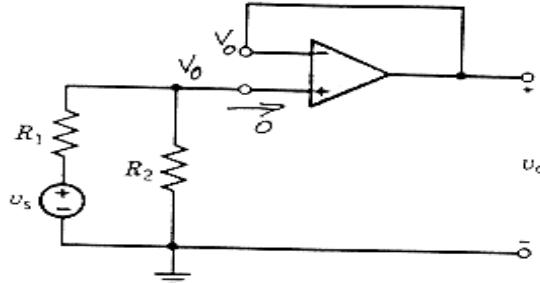
$$v_a = \frac{R_2}{R_1 + R_2} v_s$$

$$\frac{v_a}{R_3} + \frac{v_a - v_0}{R_4} + 0 = 0$$

$$\frac{v_0}{v_a} = 1 + \frac{R_4}{R_3} \Rightarrow \frac{v_0}{v_s} = \left(\frac{R_2}{R_1 + R_2} \right) \left(1 + \frac{R_4}{R_3} \right)$$

b) When $R_2 \gg R_1$ then $\frac{R_2}{R_1 + R_2} \approx \frac{R_2}{R_2} = 1$ and $\frac{v_0}{v_s} \approx 1 + \frac{R_4}{R_3}$

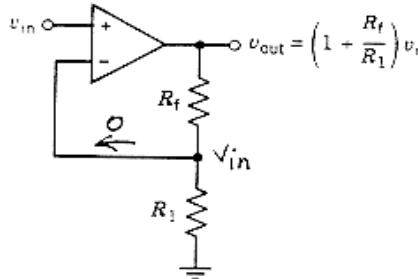
Ex. 6.5-1



$$\frac{v_0}{R_2} + \frac{v_0 - v_s}{R_1} + 0 = 0$$

$$\frac{v_0}{v_s} = \frac{R_2}{R_1 + R_2}$$

Ex. 6.6-1



$$\frac{v_0 - v_{out}}{R_f} + \frac{v_{in}}{R_1} + 0 = 0 \Rightarrow v_{out} = \left(1 + \frac{R_f}{R_1} \right) v_{in}$$

when $R_f = 100k\Omega$ and $R_1 = 25k\Omega$ then

$$\frac{v_{out}}{v_{in}} = \left(1 + \frac{100 \cdot 10^3}{25 \cdot 10^3} \right) = 5$$

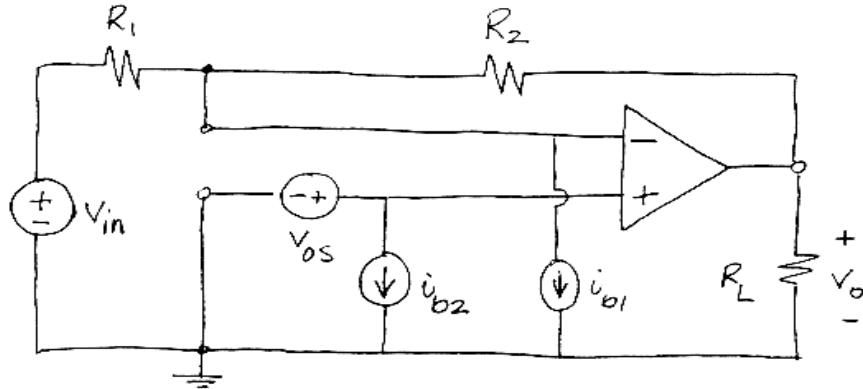
(b) Noninverting amplifier

Ex. 6.7-1 Analysis of the circuit in Section 6.7 showed that output offset voltage = $6 v_{os} + (50 \cdot 10^3) i_{b1}$

For a μ A741 opamp, $|v_{os}| \leq 1mV$ and $|i_{b1}| \leq 80nA$ so

$$|\text{output offset voltage}| = |6 v_{os} + (50 \cdot 10^3) i_{b1}| \leq 6 (10^{-3}) + (50 \cdot 10^3)(80 \cdot 10^{-9}) = 10mV$$

Ex. 6.7-2



$$V_o = -\frac{R_2}{R_1} V_{in} + \left(1 + \frac{R_2}{R_1}\right) v_{os} + R_2 i_{b1}$$

When $R_2 = 10k\Omega$, $R_1 = 2k\Omega$, $|v_{os}| \leq 5mV$ and $|i_{b1}| \leq 500nA$ then
output offset voltage $\leq 6(5 \cdot 10^{-3}) + (10 \cdot 10^3)(500 \cdot 10^{-9}) \leq 35 \cdot 10^{-3} = 35mV$

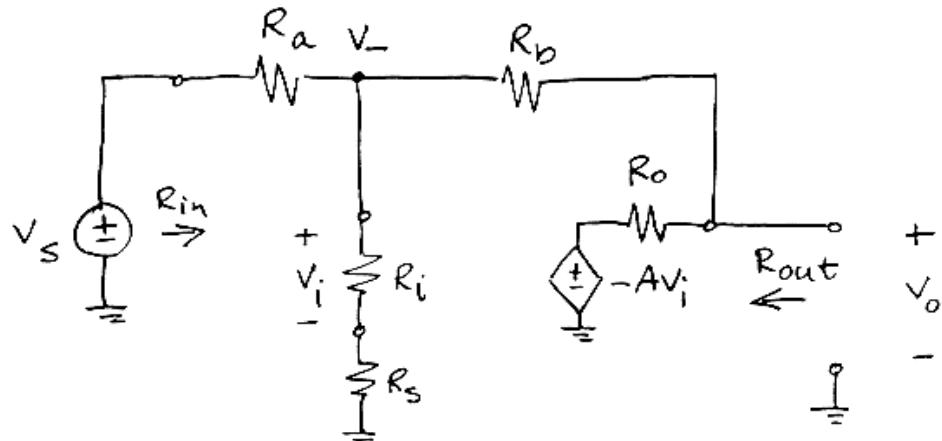
Ex. 6.7-3

Analysis of this circuit in Section 6.7 showed that output offset voltage $= 6v_{os} + (50 \times 10^3)i_{b1}$

For a typical OPA1O1AM, $v_{os} = 0.1mV$ and $i_b = 0.012nA$ so

$$\begin{aligned} |\text{output offset voltage}| &\leq 6[0.1 \times 10^{-3}] + (50 \times 10^3)[0.012 \times 10^{-9}] \\ &\leq 0.6 \times 10^{-3} + 0.6 \times 10^{-6} \approx 0.6 \times 10^{-3} \\ &\leq 0.6mV \end{aligned}$$

Ex. 6.7-4



$$\frac{V_- - V_s}{R_a} + \frac{V_- - V_o}{R_b} + \frac{V_-}{R_i + R_s} = 0$$

$$\frac{V_o - \left(-A \frac{R_i}{R_i + R_s} V_- \right)}{R_0} + \frac{V_o - V_-}{R_b} = 0$$

After some algebra

$$A_v = \frac{V_o}{V_s} = \frac{R_0(R_i + R_s) + A R_i R_f}{(R_f + R_0)(R_i + R_s) + R_a(R_f + R_0 + R_i + R_s) - A R_i R_a}$$

For the given values, $A_v = -2.00006$

Ex. 6.8-1

```

Spice deck
V1      1 0      200mV
V2      2 0      125mV
V3      3 0      250mV
R1      1 5      50k
R2      2 5      25k
R3      5 4      100k
XOA1    5 0 4    IDEAL_OP_AMP
R4      4 6      25k
R5      3 6      10k
R6      6 7      50k
R7      7 0      100k
XOA2    6 0 7    IDEAL_OP_AMP

.SUBCKT IDEAL_OP_AMP 1 2 3
E 3 0 1 2 -1G
.ENDS IDEAL_OP_AMP
.END

NODE      VOLTAGE      NODE      VOLTAGE      NODE      VOLTAGE
( 1)      .2000       ( 2)      .1250       ( 3)      .2500
( 4)      -.9000      ( 5)      900.0E-12   ( 6)      -550.0E-12
( 7)      .5500

```

Ex. 6.8-2

```

Spice deck
V1      1 0      200mV
V2      2 0      125mV
V3      3 0      250mV
R1      1 5      25k
R2      2 5      50k
R3      5 4      100k
XOA1    5 0 4    TL501_OP_AMP
R4      4 6      25k
R5      3 6      10k
R6      6 7      50k
R7      7 0      100k
XOA2    6 0 7    TL501_OP_AMP

.SUBCKT TL501_OP_AMP 1 2 5
IB1     1 0      .0175nA
IB2     2 0      .0425nA
VOS     3 2      .59mV
RI      1 3      1MEG
E       4 0 1 3  -105000
RO      4 5      250
.ENDS TL501_OP_AMP
.END

NODE      VOLTAGE      NODE      VOLTAGE      NODE      VOLTAGE
( 1)      .2000       ( 2)      .1250       ( 3)      .2500
( 4)      -.8958      ( 5)      598.6E-06   ( 6)      584.8E-06
( 7)      .5463       (XOA1.3)  590.0E-06   (XOA1.4)  -.9070
(XOA2.3)  590.0E-06  (XOA2.4)  .5504

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Ex. 6.8-3

```

Spice deck
V1    1  0          200mV
V2    2  0          125mV
V3    3  0          250mV
R1    1  5          5k
R2    2  5          2.5k
R3    5  4          10k
XOA1  5  0  4      UA741_OP_AMP
R4    4  6          2.5k
R5    3  6          1k
R6    6  7          5k
R7    7  0          10k
XOA2  6  0  7      UA741_OP_AMP

SUBCKT UA741_OP_AMP 1 2 5
IB1    1  0          70nA
IB2    2  0          90nA
VOS    3  2          1mV
RI     1  3          2MEG
E      4  0  1  3   -200000
RO     4  5          75
.ENDS  UA741_OP_AMP
.END

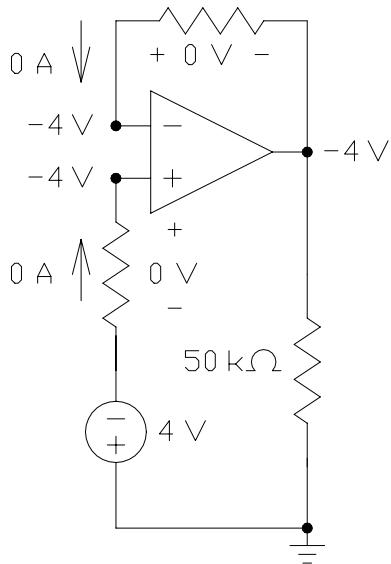
NODE      VOLTAGE      NODE      VOLTAGE      NODE      VOLTAGE
( 1)      .2000       ( 2)      .1250       ( 3)      .2500
( 4)      -.8958      ( 5)      .0010       ( 6)      997.2E-06
( 7)      .5429       (XOA1.3)  .0010       (XOA1.4)  -.9258
(XOA2.3)  .0010       (XOA2.4)  .5551

```

Problems

Section 6-4: The Ideal Operational Amplifier

P6.4-1



P6.4-2

KVL at:

$$-12 + 3000i + 0 + 2000i = 0$$

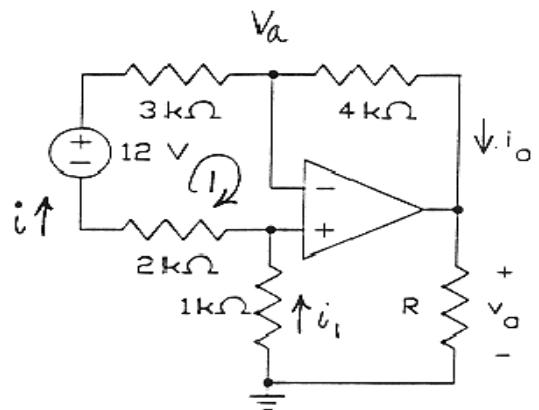
$$\Rightarrow i = \frac{12}{5000} = 2.4 \text{ mA}$$

$$i_0 = i = 2.4 \text{ mA}$$

$$i_1 = i = 2.4 \text{ mA}$$

$$v_a = -i_1(1000) + 0 \\ = -2.4 \text{ V}$$

$$v_0 = v_a - i_0(4000) = -2.4 - (2.4 \times 10^{-3})(4000) = -12 \text{ V}$$



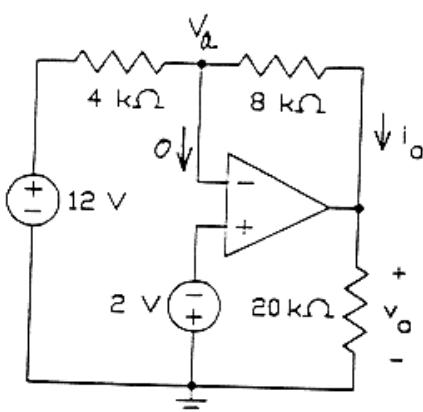
P6.4-3

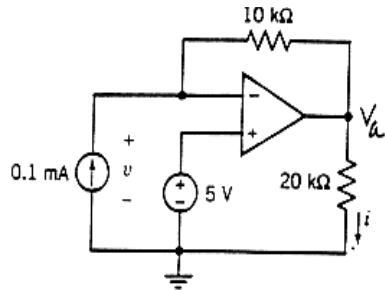
$$v_a = -2 \text{ V}$$

$$\frac{v_0 - (-2)}{8000} + \frac{12 - (-2)}{4000} = 0$$

$$\Rightarrow v_0 = -30 \text{ V}$$

$$i_0 = \frac{-2 - v_0}{8000} = 3.5 \text{ mA}$$



P6.4-4

$$v = 5 \text{ V}$$

$$-\left(\frac{v_a - 5}{10000}\right) - 0.1 \times 10^{-3} - 0 = 0$$

$$\Rightarrow v_a = 4 \text{ V}$$

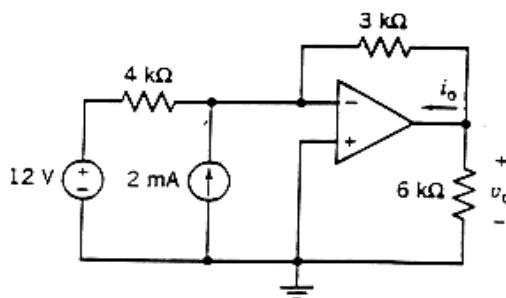
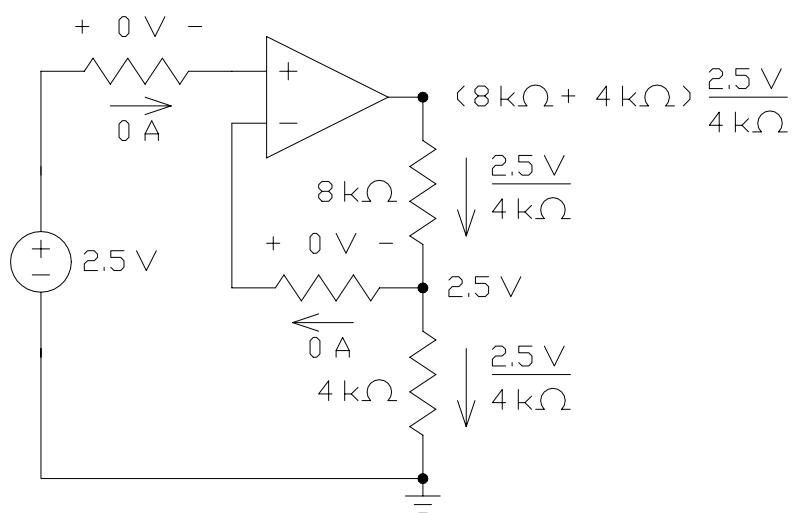
$$i = \frac{v_a}{20000} = \frac{1}{5} \text{ mA}$$

P6.4-5

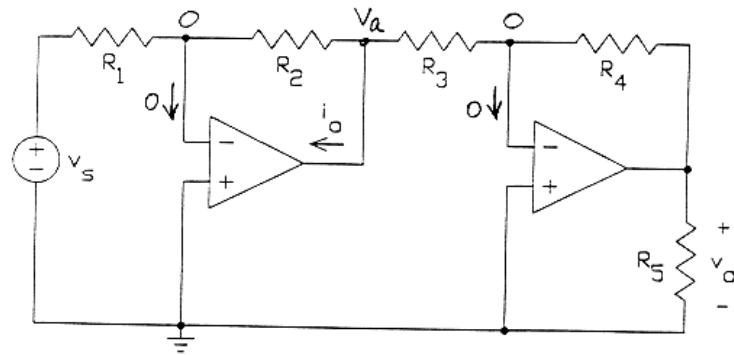
$$-\left(\frac{v_0 - 0}{3000}\right) - \left(\frac{12 - 0}{4000}\right) - 2 \cdot 10^{-3} = 0$$

$$\Rightarrow v_0 = -15 \text{ V}$$

$$i_0 + \frac{v_0}{6000} + \frac{v_0}{3000} = 0 \Rightarrow i_0 = 7.5 \text{ mA}$$

**P6.4-6**

P6.4-7

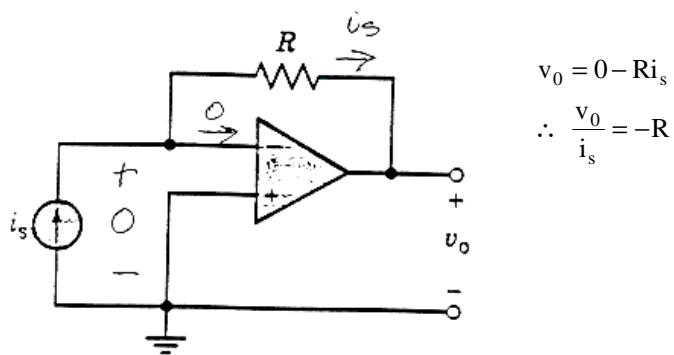


$$-\left(\frac{v_s - 0}{R_1}\right) - \left(\frac{v_a - 0}{R_2}\right) + 0 = 0 \Rightarrow v_a = -\frac{R_2}{R_1} v_s$$

$$i_0 = \frac{0 - v_a}{R_2} + \frac{0 - v_a}{R_3} = \frac{R_2 + R_3}{R_2 R_3} v_a = -\left(\frac{R_2 + R_3}{R_1 R_3}\right) v_s$$

$$-\left(\frac{v_0 - 0}{R_4}\right) - \left(\frac{v_a - 0}{R_3}\right) + 0 = 0 \Rightarrow v_0 = -\frac{R_4}{R_3} v_a = \frac{R_2 R_4}{R_1 R_3} v_s$$

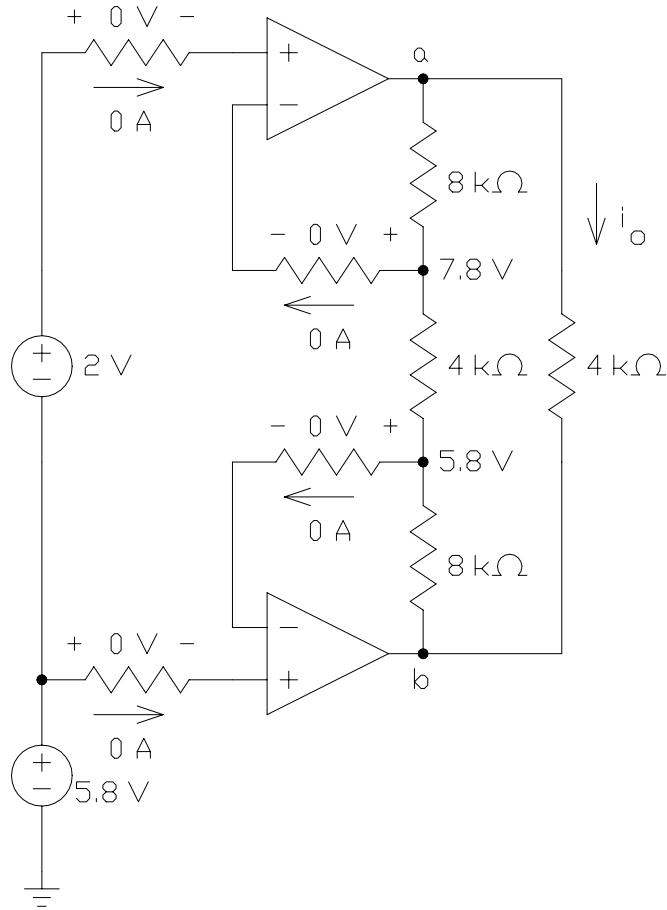
P6.4-8



$$v_0 = 0 - R i_s$$

$$\therefore \frac{v_0}{i_s} = -R$$

P6.4-9



P 6.4-10

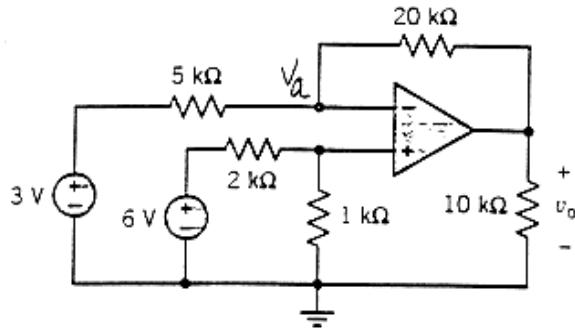
$$\text{KCL at node a: } \frac{v_a - (-18)}{4000} + \frac{v_a}{8000} + 0 = 0 \Rightarrow v_a = -12 \text{ V}$$

The node voltages at the input nodes of ideal op amps are equal so $v_b = v_a$.

$$\text{Voltage division: } v_o = \frac{8000}{4000 + 8000} v_b = -8 \text{ V}$$

Section 6-5: Nodal Analysis of Circuits Containing Ideal Operational Amplifiers

P6.5-1



$$v_a = \frac{1000}{2000+1000} 6 = 2 \text{ V}$$

$$-\left(\frac{v_0 - 2}{20000}\right) - \left(\frac{3 - 2}{5000}\right) = 0$$

$$\Rightarrow v_0 = -2 \text{ V}$$

P6.5-2

$$\text{KCL at node b: } \frac{v_b - 2}{20e3} + \frac{v_b}{40e3} + \frac{v_b + 5}{40e3} = 0 \Rightarrow v_b = -\frac{1}{4} \text{ V}$$

$v_e = v_b = -\frac{1}{4} \text{ V}$ because the node voltages at the input nodes of an ideal op amp are equal

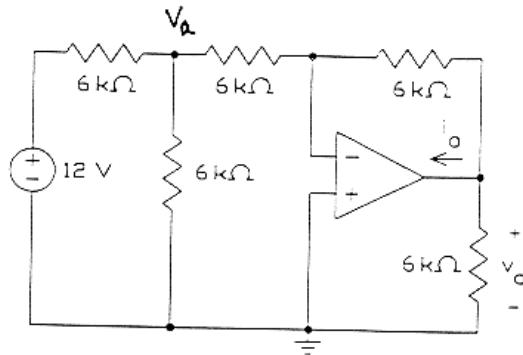
$$\text{KCL at node e: } \frac{v_e}{1000} + \frac{v_d + v_e}{40e3} = 0 \Rightarrow v_d = 10v_e = -\frac{10}{4} \text{ V}$$

P6.5-3

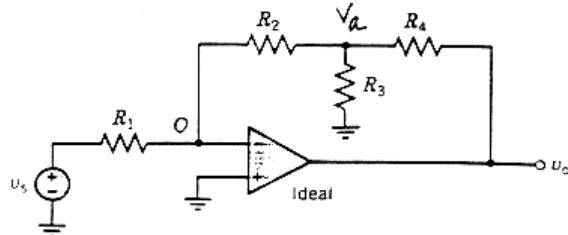
$$0 = \frac{v_a - 12}{6000} + \frac{v_a}{6000} + \frac{v_a - 0}{6000} \Rightarrow v_a = 4 \text{ V}$$

$$-\left(\frac{v_a - 0}{6000}\right) + 0 + \left(\frac{0 - v_0}{6000}\right) = 0 \Rightarrow v_0 = -v_a = -4 \text{ V}$$

$$i_0 = -\left(\frac{0 - v_0}{6000}\right) + \frac{v_0}{6000} = 0 \Rightarrow i_0 = -\frac{v_0}{3000} = 1.33 \text{ mA}$$



P6.5-4



$$-\left(\frac{v_a - 0}{R_2}\right) - \left(\frac{v_s - 0}{R_1}\right) = 0$$

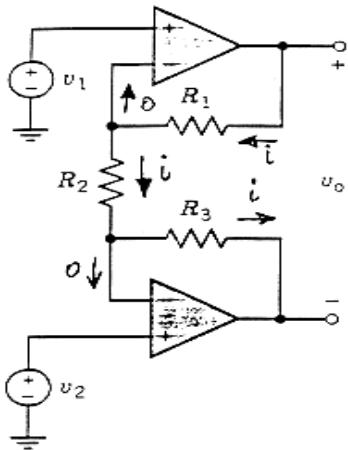
$$\Rightarrow v_a = -\frac{R_2}{R_1} v_s$$

$$\frac{v_a - v_0}{R_4} + \frac{v_a}{R_3} + \frac{v_a - 0}{R_2} = 0 \Rightarrow v_0 = R_4 \left(\frac{1}{R_4} + \frac{1}{R_3} + \frac{1}{R_2} \right) v_a = \frac{R_2 R_3 + R_2 R_4 + R_3 R_4}{R_2 R_3} v_a$$

$$= -\frac{R_2 R_3 + R_2 R_4 + R_3 R_4}{R_1 R_3} v_s$$

Plug in values \Rightarrow yields $\frac{v_0}{v_s} = -200$

P6.5-5



$$i = \frac{v_1 - v_2}{R_2}$$

$$v_0 = (R_1 + R_2 + R_3)i$$

$$= \frac{R_1 + R_2 + R_3}{R_2} (v_1 - v_2)$$

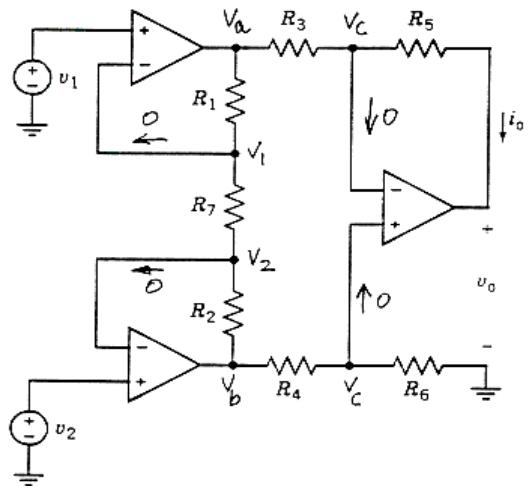
P6.5-6

$$\frac{v_1 - v_a}{R_1} + \frac{v_1 - v_2}{R_7} + 0 = 0 \Rightarrow v_a = \left(1 + \frac{R_1}{R_7}\right) v_1 - \frac{R_1}{R_7} v_2$$

$$\frac{v_2 - v_b}{R_2} - \frac{v_1 - v_2}{R_7} + 0 = 0 \Rightarrow v_b = \left(1 + \frac{R_2}{R_7}\right) v_2 - \frac{R_2}{R_7} v_1$$

$$-\left(\frac{v_b - v_c}{R_4}\right) + \frac{v_c - 0}{R_6} + 0 = 0 \Rightarrow v_c = \frac{R_6}{R_4 + R_6} v_b$$

$$-\left(\frac{v_a - v_c}{R_3}\right) + \left(\frac{v_c - v_0}{R_5}\right) + 0 = 0 \Rightarrow v_0 = -\frac{R_5}{R_3} v_a + \left(1 + \frac{R_5}{R_3}\right) v_c$$



P6.5-8

$$-\left(\frac{v_a - 0}{10000}\right) + 0 - \left(\frac{(v_a + 6) - 0}{30000}\right) = 0$$

$$\Rightarrow v_a = -1.5 \text{ V}$$

$$\frac{v_a - 0}{10000} + \frac{v_a + 6 - 0}{30000} + \frac{v_a - v_b}{30000} + \frac{(v_a + 6) - 0}{10000} = 0$$

$$\Rightarrow 3v_a + v_a + 6 + v_a - v_b + 3[(v_a + 6) - v_b] = 0$$

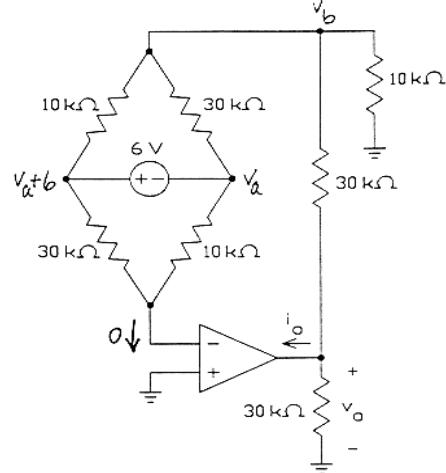
$$\Rightarrow v_b = 2v_a + 6 = 3 \text{ V}$$

$$\frac{v_b}{10000} + \frac{v_b - v_0}{30000} - \left(\frac{v_a - v_b}{30000}\right) - \left(\frac{(v_a + 6) - v_b}{10000}\right) = 0$$

$$\Rightarrow 3v_b + (v_b - v_0) - (v_a - v_b) - 3[(v_a + 6) - v_b] = 0$$

$$\Rightarrow v_0 = 8v_b - 4v_a - 18 = 12 \text{ V}$$

$$i_0 + \frac{v_0}{30000} + \frac{v_0 - v_b}{30000} = 0 \Rightarrow i_0 = -0.7 \text{ mA}$$



P6.5-9

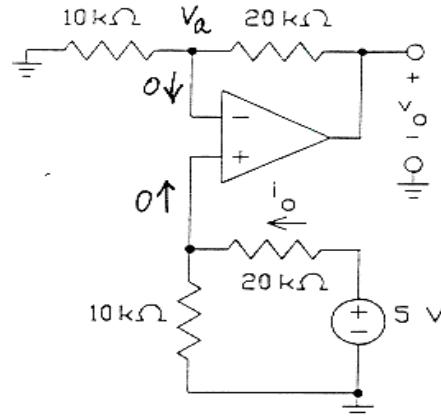
$$-i_0(10000) - i_0(20000) + 5 = 0$$

$$i_0 = \frac{1}{6} \text{ mA}$$

$$v_a = 10000 \quad i_0 = \frac{10}{6} \text{ V}$$

$$\frac{v_a}{10000} + \frac{v_a - v_0}{20000} = 0$$

$$\Rightarrow v_0 = 3v_a = 5 \text{ V}$$



P6.5-10

$$\text{KCL at node b: } \frac{v_b + 12}{40e3} + \frac{v_b}{20e3} = 0 \Rightarrow v_b = -4 \text{ V}$$

$v_c = v_b = -4 \text{ V}$ because the node voltages at the input nodes of an ideal op amp are equal.

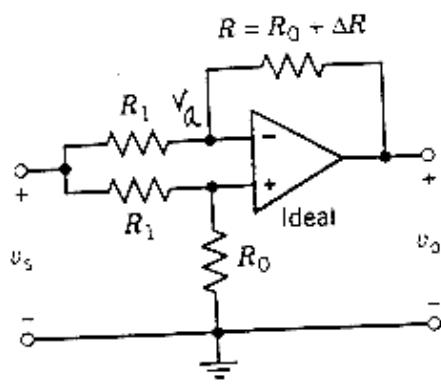
$v_d = v_c + 0 \cdot 10e3 = -4 \text{ V}$ because the currents into the inputs of an ideal op amp are zero.

$$\text{KCL at node g: } -\left(\frac{v_f - v_g}{20e3}\right) + \frac{v_g}{40e3} = 0 \Rightarrow v_g = \frac{2}{3}v_f$$

$$\text{KCL at node d: } \frac{v_d - v_f}{20e3} + \frac{v_d - \frac{2}{3}v_f}{20e3} = 0 \Rightarrow v_f = \frac{6}{5}v_d = -\frac{24}{5} \text{ V so } v_g = \frac{2}{3}v_f = -\frac{15}{5} \text{ V.}$$

$v_c = v_g = -\frac{16}{5} \text{ V}$ because the node voltages at the input nodes of an ideal op amp are equal

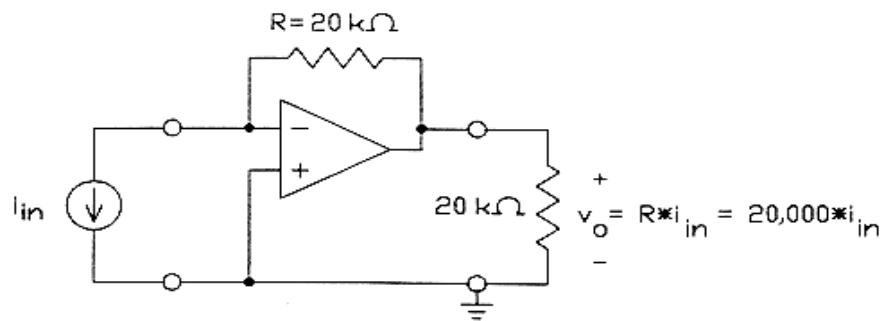
P6.5-11



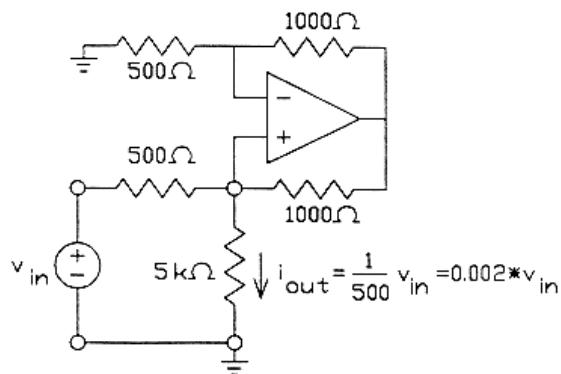
$$\begin{aligned}
 v_a &= \frac{R_0}{R_1 + R_0} v_s \\
 \frac{v_a - v_s}{R_1} + \frac{v_a - v_0}{R_0 + \Delta R} &= 0 \\
 \frac{R_0 + \Delta R}{R_1} (v_a - v_s) + v_a &= v_0 \\
 v_0 &= \left[\left(\frac{R_0 + \Delta R}{R_1} + 1 \right) \frac{R_0}{R_1 + R_0} - \frac{R_0 + \Delta R}{R_1} \right] v_s \\
 &= -\frac{\Delta R}{R_1 + R_0} v_s \\
 &= \left(-v_s \frac{R_0}{R_1 + R_0} \right) \frac{\Delta R}{R_0}
 \end{aligned}$$

Section 6-6: Design Using Operational Amplifier

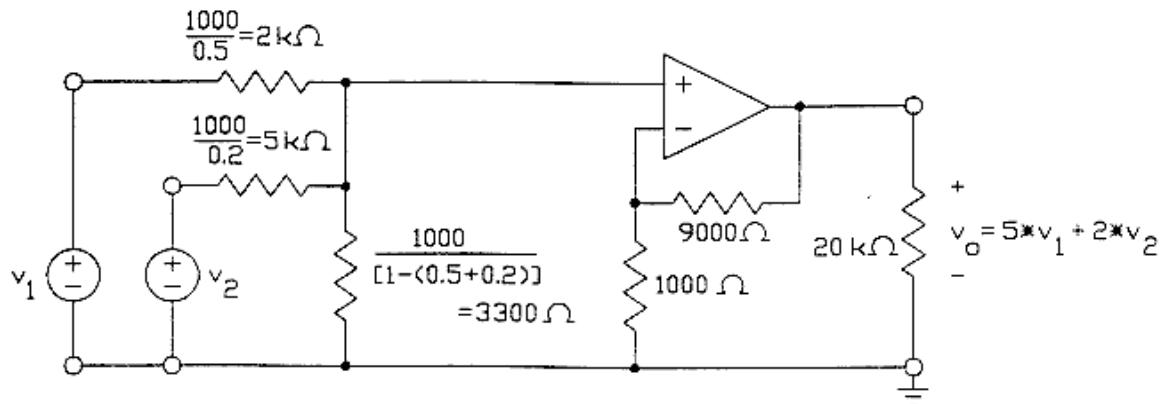
P6.6-1



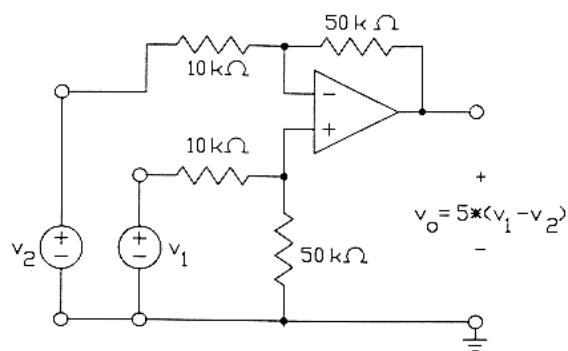
P6.6-2



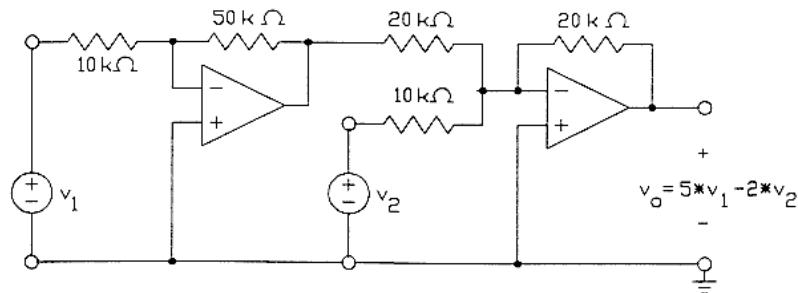
P6.6-3



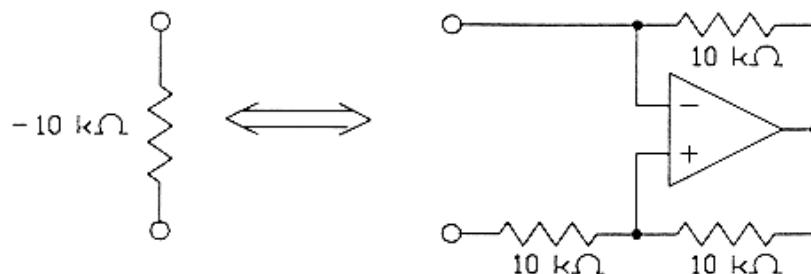
P6.6-4



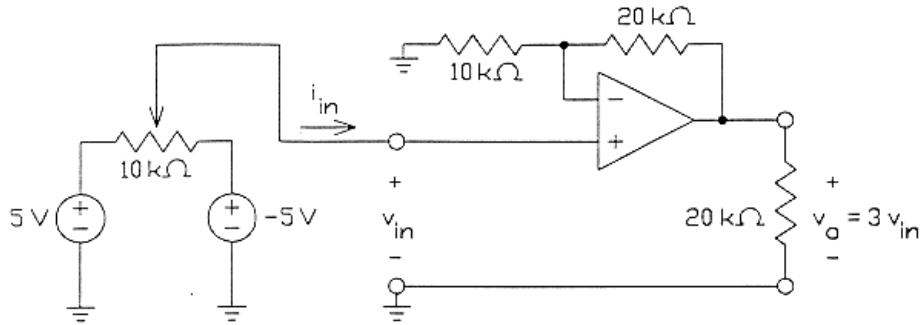
P6.6-5



P6.6-6



P6.6-7



P6.6-8

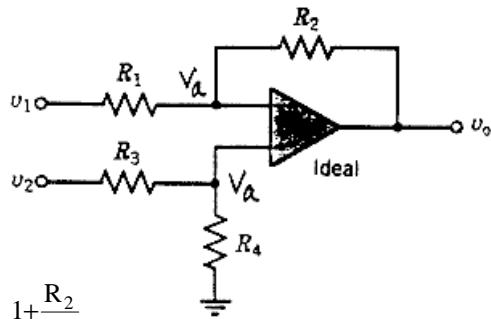
$$(a) \quad v_a = \frac{R_4}{R_3 + R_4} v_2$$

$$\frac{v_a - v_1}{R_1} + \frac{v_a - v_0}{R_2} = 0$$

$$\Rightarrow v_0 = \left(\frac{R_2}{R_1} + 1 \right) v_a - \frac{R_2}{R_1} v_1$$

$$= \left(\frac{R_2}{R_1} + 1 \right) \frac{R_4}{R_3 + R_4} v_2 - \frac{R_2}{R_1} v_1 = \frac{1 + \frac{R_2}{R_1}}{1 + \frac{R_3}{R_4}} v_2 - \frac{R_2}{R_1} v_1$$

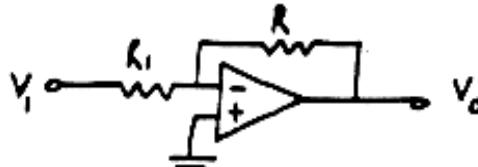
$$(b) \quad 11 = \frac{R_2}{R_1} \text{ and } 4 = \frac{1 + \frac{R_2}{R_1}}{1 + \frac{R_3}{R_4}} = \frac{12}{1 + \frac{R_3}{R_4}} \Rightarrow \frac{R_3}{R_4} = 2$$



For example:

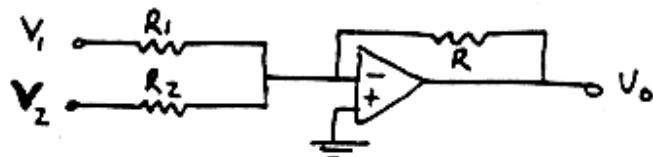
$$R_1 = 10\text{k}\Omega, \quad R_2 = 110\text{k}\Omega, \quad R_3 = 20\text{k}\Omega \quad \& \quad R_4 = 10\text{k}\Omega$$

P6.6-9



We know this ckt yields $v_0 = -\frac{R}{R_1}v_1$

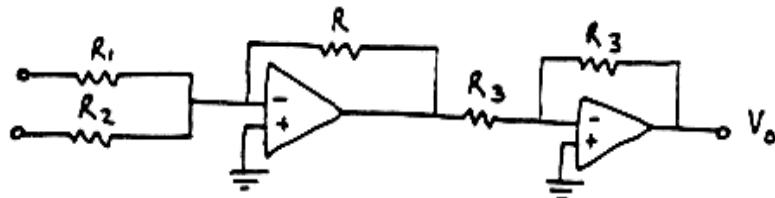
so consider adding another input lead into the negative terminal



with $v_i \approx 0$, KCL at negative input yields

$$-\frac{v_1}{R_1} - \frac{v_2}{R_2} = \frac{v_0}{R} \Rightarrow v_0 = -\frac{R}{R_1}v_1 - \frac{R}{R_2}v_2$$

Since need to invert the answer, add an inverter to the output,

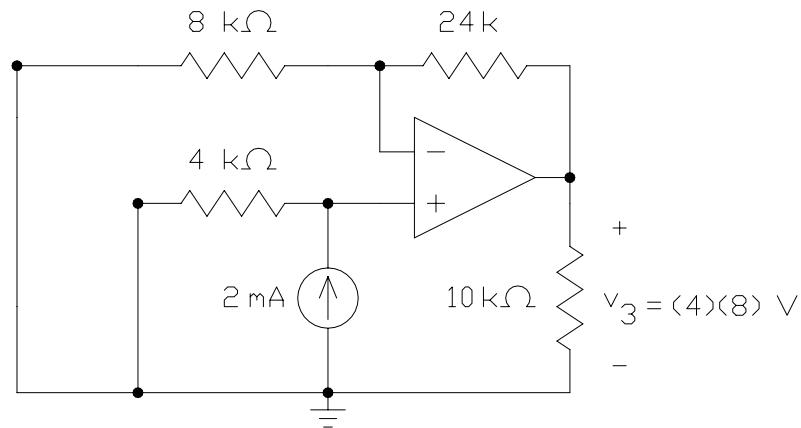
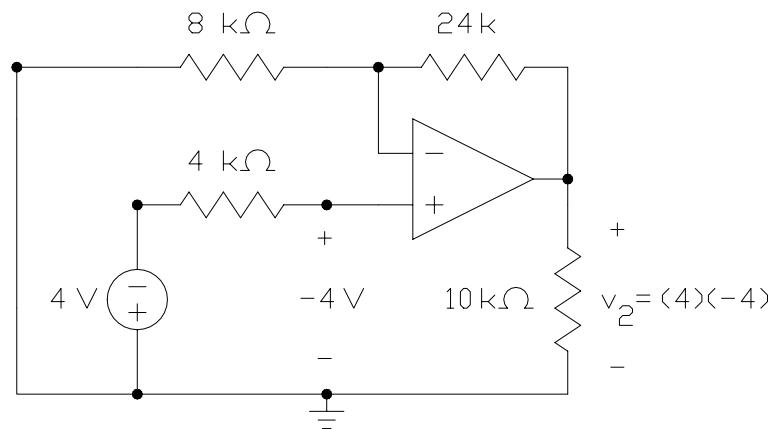
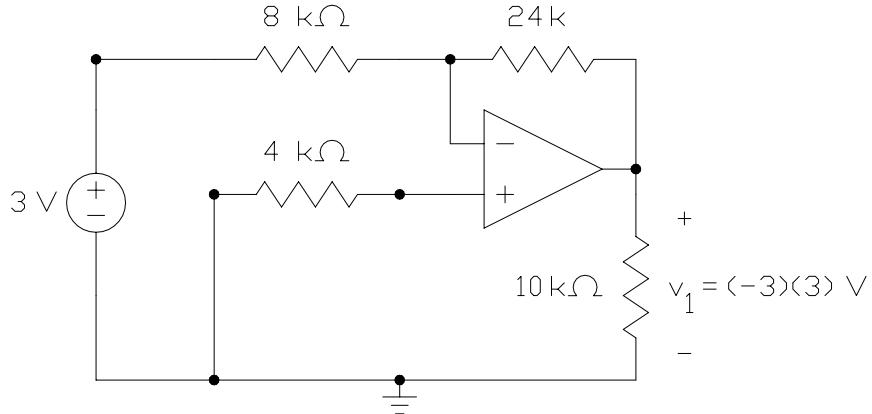


Let $R_3 = 10k\Omega$

$$\text{Now } \frac{R}{R_1} = 6 \text{ and } \frac{R}{R_2} = 2 \quad \therefore \text{let } R = 60 k\Omega$$

$$R_1 = 10k\Omega \text{ and } \underline{R_2 = 30k\Omega}$$

P6.6-10

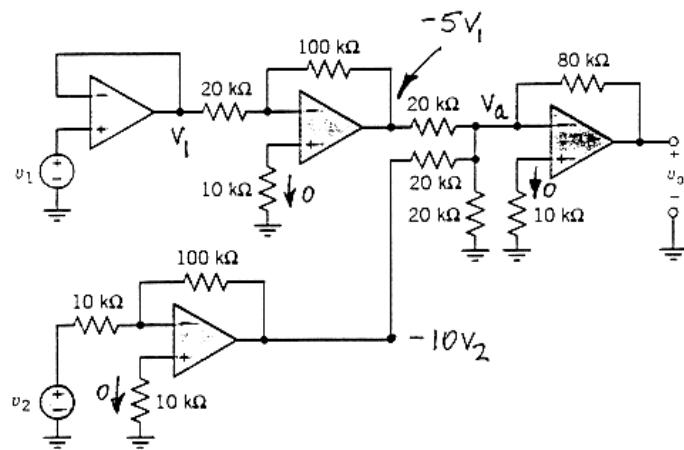


Using superposition, $v_o = v_1 + v_2 + v_3 = -9 - 16 + 32 = 7 \text{ V}$

P6.6-11

R_i	6	12	24	6 12	6 24	12 12	12 24	6 12 12	6 12 24	12 12 24
R_o	12 12 24	6 12 24	6 12 12	12 24	12 12	6 24	6 12	24	12	6
$-v_o/v_s$	0.8	0.286	0.125	2	1.25	0.8	0.5	8	3.5	1.25

P6.6-12



$$\frac{v_a - (-5v_1)}{20k\Omega} + \frac{v_a - (-10v_2)}{20k\Omega} + \frac{v_a}{20k\Omega} + \frac{v_a - v_0}{80k\Omega} = 0$$

$$4(v_a + 5v_1) + 4(v_a + 10v_2) + 4v_a + v_a - v_0 = 0$$

$$v_a = 0(10k\Omega) = 0$$

$$v_0 = 20v_1 + 40v_2$$

P 6.6-13

$$v_c = -(3v_a + 1)$$

$$v_o = -(3v_b + 2v_c + 2) = -(3v_b - 2(3v_a + 1) + 2) = -3v_b + 6v_a$$

P6.6-14

(a) $2R \parallel 2R = R$

R in series with R = 2R

⋮

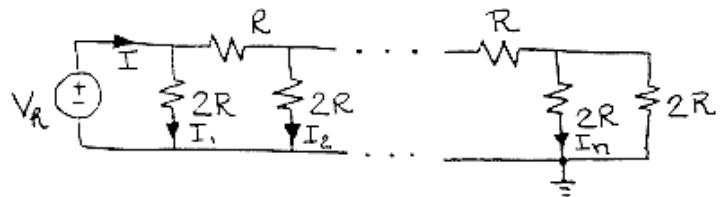
$$R_{eq} = R$$

$$\underline{I = \frac{V_R}{R}}, \quad I_1 = \frac{V_R}{R} \left(\frac{2R}{4R} \right) = \frac{V_R}{2R}$$

$$I_2 = \frac{V_R}{2R} \left(\frac{2R}{4R} \right) = \frac{V_R}{2^2 R}$$

⋮

$$I_n = \frac{V_R}{2^n R}$$



(b) $\frac{V_0}{R_f} = b_1 I_1 + b_2 I_2 + \dots + b_n I_n$

$$V_0 = R_f \left[b_1 \frac{V_R}{2R} + b_2 \frac{V_R}{2^2 R} + \dots + b_n \frac{V_R}{2^n R} \right] = \frac{R_f V_R}{R} \left[b_1 2^{-1} + b_2 2^{-2} + \dots + b_n 2^{-n} \right]$$

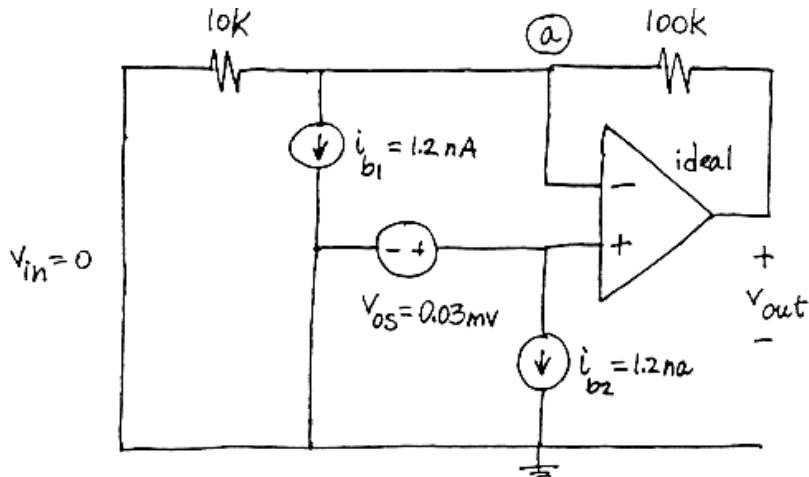
$$\begin{aligned} (c) \quad I^+ + I^- &= \frac{V_R}{2R} + \frac{V_R}{2^2 R} + \dots + \frac{V_R}{2^n R} \\ &= \frac{V_R}{R} \left[2^{-1} + 2^{-2} + \dots + 2^{-n} \right] \\ &= \frac{V_R}{R} \left[1 - 2^{-n} \right] \end{aligned}$$

(d)

$-V_0$	b1	b2	b3	b4
0	0	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
8	0	0	0	0
9	1	0	0	1
10	1	0	1	0
11	1	0	1	1
12	1	1	0	0
13	1	1	0	1
14	1	1	1	0
15	1	1	1	1

Section 6-7: Characteristics of the Practical Operational Amplifier

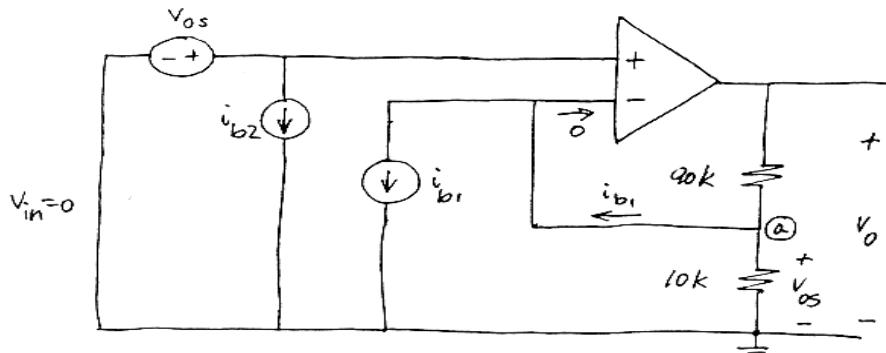
P6.7-1



$$\text{The node equation at node } a \text{ is } \frac{v_{out} - v_{os}}{100k} = \frac{v_{os}}{10k} + i_{b1}$$

$$\begin{aligned} \text{so } v_{out} &= \left(1 + \frac{100k\Omega}{10k\Omega}\right)v_{os} + 100k\Omega \cdot i_{b1} = 11v_{os} + 100k\Omega \cdot i_{b1} \\ &= 11(0.03 \text{ mV}) + 100k\Omega(1.2 \text{ nA}) \\ &= \underline{0.45 \text{ mV}} \end{aligned}$$

P6.7-2



The node equation at node a is

$$\frac{v_{os}}{10k} + i_{b1} = \frac{v_o - v_{os}}{90k\Omega}$$

$$\begin{aligned} \text{so } v_o &= \left(1 + \frac{90k\Omega}{10k\Omega}\right)v_{os} + 90k\Omega i_{b1} = 10v_{os} + 90k\Omega i_{b1} \\ &= 10(5 \text{ mV}) + 90k\Omega(0.05 \text{ nA}) \\ &= 50.0045 \text{ mV} \approx \underline{50 \text{ mV}} \end{aligned}$$