Image Enhancement in the frequency domain

Inel 5046

Prof. Vidya Manian

Introduction

- 2D Fourier transform
- Basics of filtering in frequency domain
- Ideal low pass filter
- Gaussian low pass filter
- Ideal high pass filter
- Gaussian high pass filter
- FT properties and theorems

2D DFT and its inverse

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

Fourier spectrum, phase angle and power spectrum

$$|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{1/2}$$

$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

$$P(u,v) = |F(u,v)|^2$$

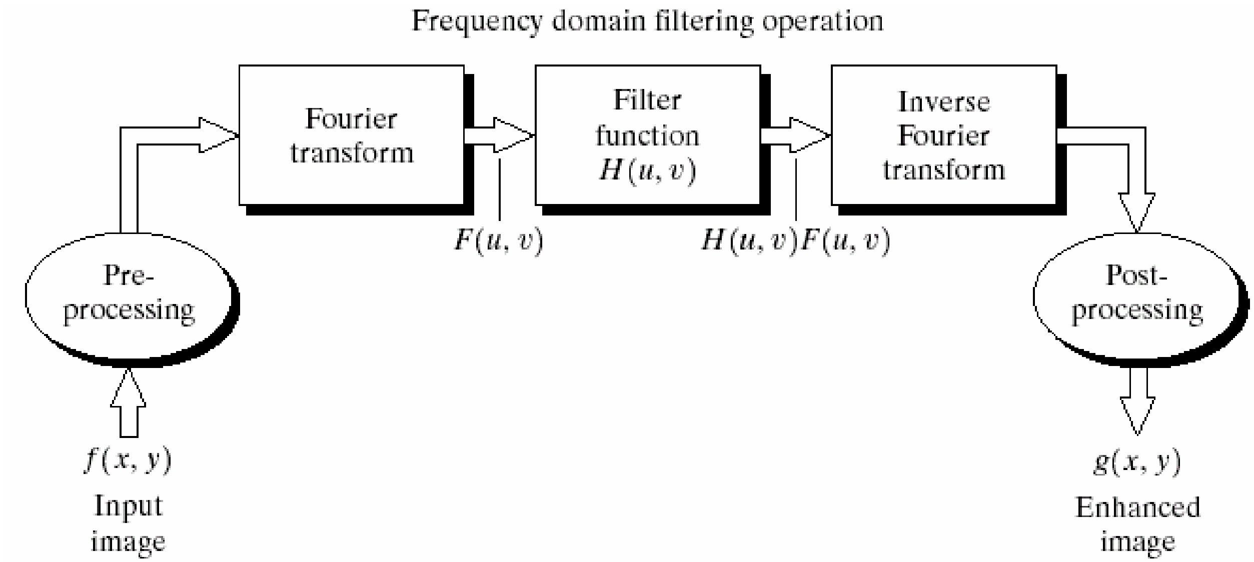
$$= R^2(u,v) + I^2(u,v)$$

Associations in frequency domain

- Each term of F(u,v) contains all values of f(x,y) modified by the values of the exponential terms
- Hard to associate specific components of image and transform
- Frequencies in FT with intensity variations in an image
- Low frequency-smooth regions, high frequency-faster grey level changes (edges and noise)

Basics of filtering in the frequency domain

- 1. Multiply the input image by $(-1)^{x+y}$ to center the transform
- 2. Compute F(u,v), the DFT of the image
- 3. Multiply F(u,v) by a filter function H(u,v)
- 4. Compute the inverse DFT of (3)
- 5. Obtain the real part of the result
- Multiply the result in (5) by $(-1)^{x+y}$



Basic steps for filtering in the frequency domain.

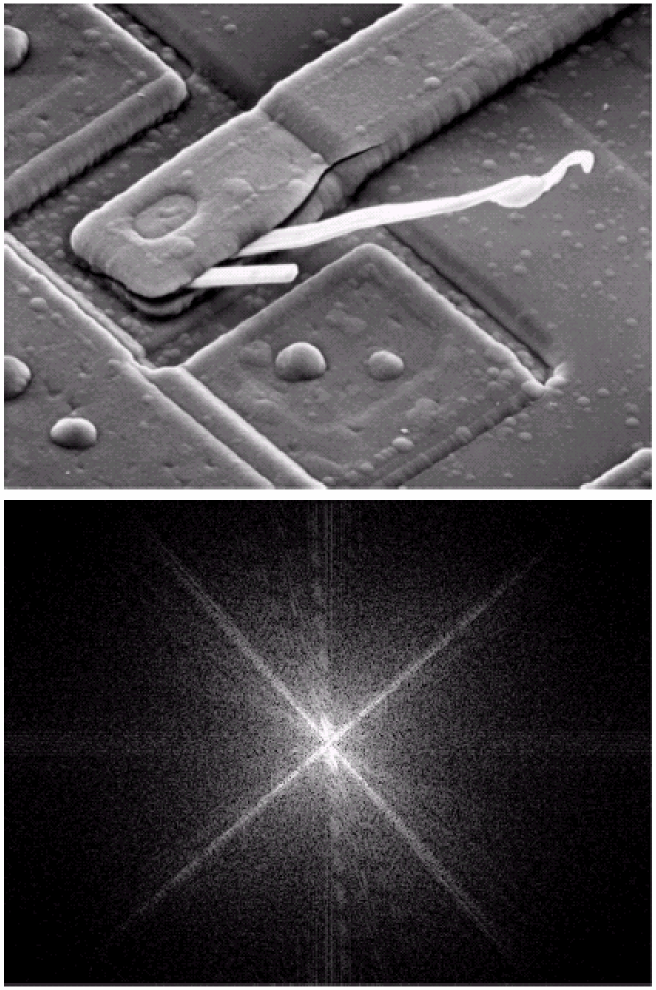
- Preprocessing-cropping image to closest even dimension, grey-level scaling, converison to floating point

Some basic filters

- F(0,0) is the dc component –the average value of the image
- Notch filter –sets F(0,0) to zero and leaves all other frequency components of the FT untouched.

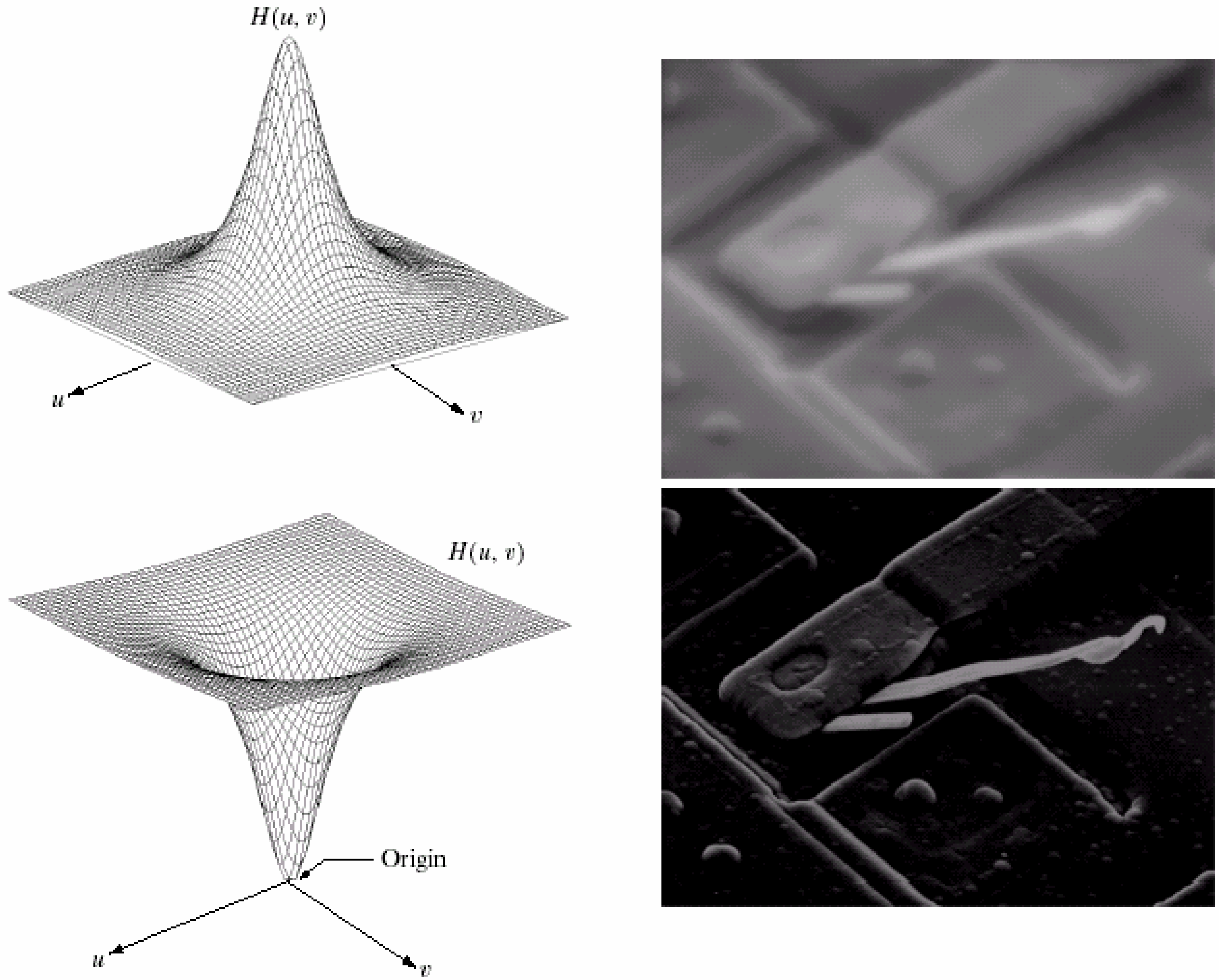
$$H(u,v) = \begin{cases} 0 & \text{if (u,v)=(M/2,N/2)} \\ 1 & \text{otherwise} \end{cases}$$

Low pass and high pass filters



a
b

(a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)



a b
c d

(a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Filters based on Gaussian functions

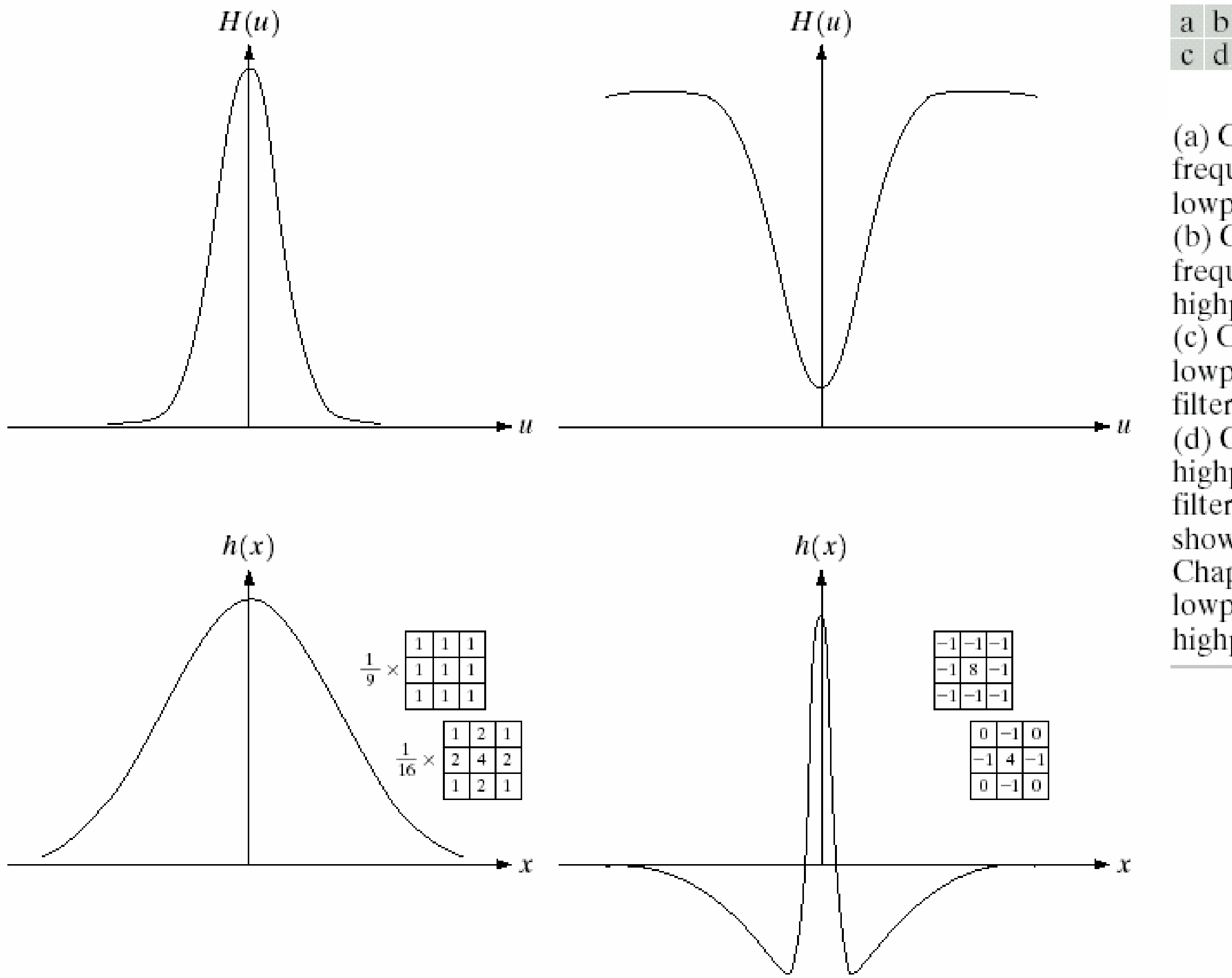
- H(u) frequency domain Gaussian filter function

$$H(u) = Ae^{-u^2/2\sigma^2}$$

- Corresponding filter in the spatial domain

$$h(x) = \sqrt{2\pi}Ae^{-2\pi^2 x^2}$$

- (1) constitute a Fourier transform pair, both components of which are Gaussian and real
- (2) These functions behave reciprocally. When H(u) has a broad profile (large value of σ), h(x) has a narrow profile, and vice versa.



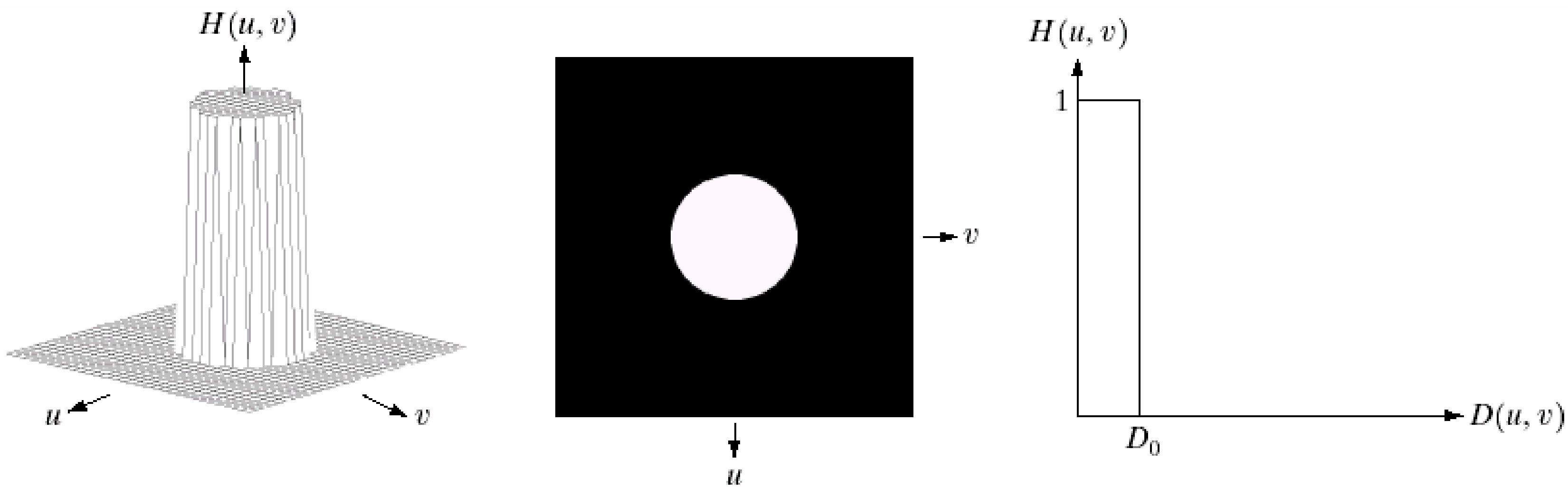
(a) Gaussian frequency domain lowpass filter. (b) Gaussian frequency domain highpass filter. (c) Corresponding lowpass spatial filter. (d) Corresponding highpass spatial filter. The masks shown are used in Chapter 3 for lowpass and highpass filtering.

Smoothing frequency domain filters Ideal low pass filter

- Cuts off all high frequency components of the Fourier transform that are at a distance greater than a specified distance D_0 from the origin of the (centered) transform

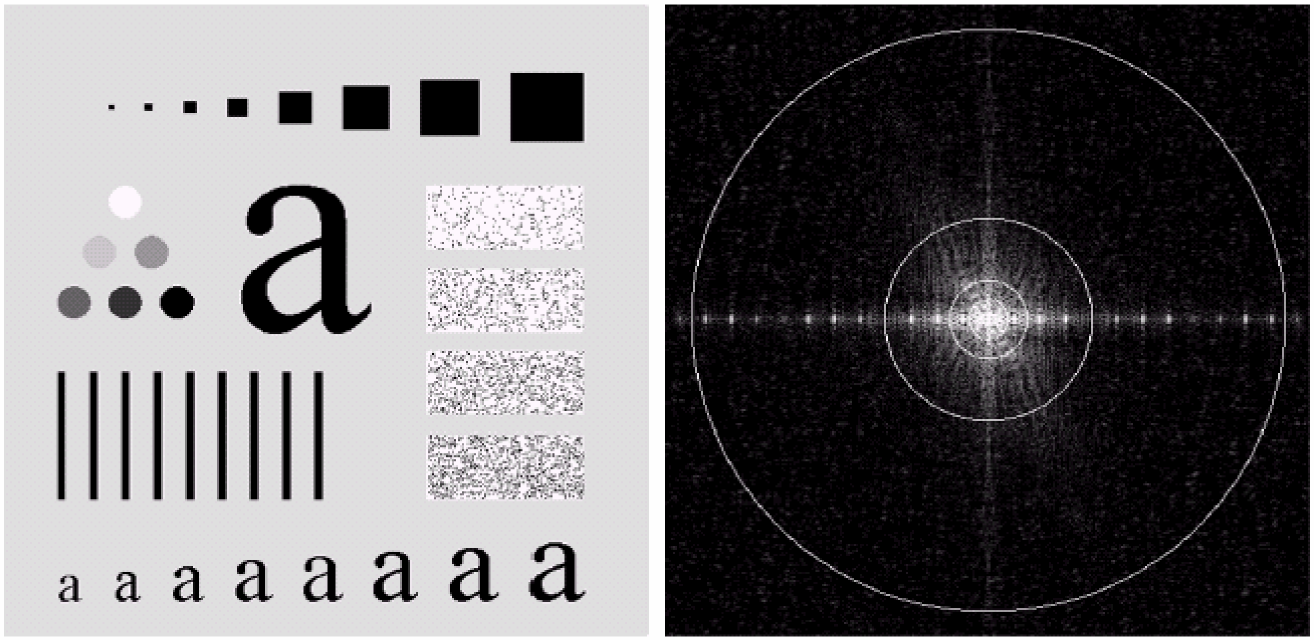
$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > 0 \end{cases}$$

- If image is of size MxN, center of the frequency rectangle is at (u,v)=(M/2,N/2)
- $D(u,v)=[(u-M/2)^2+(v-N/2)^2]^{1/2}$



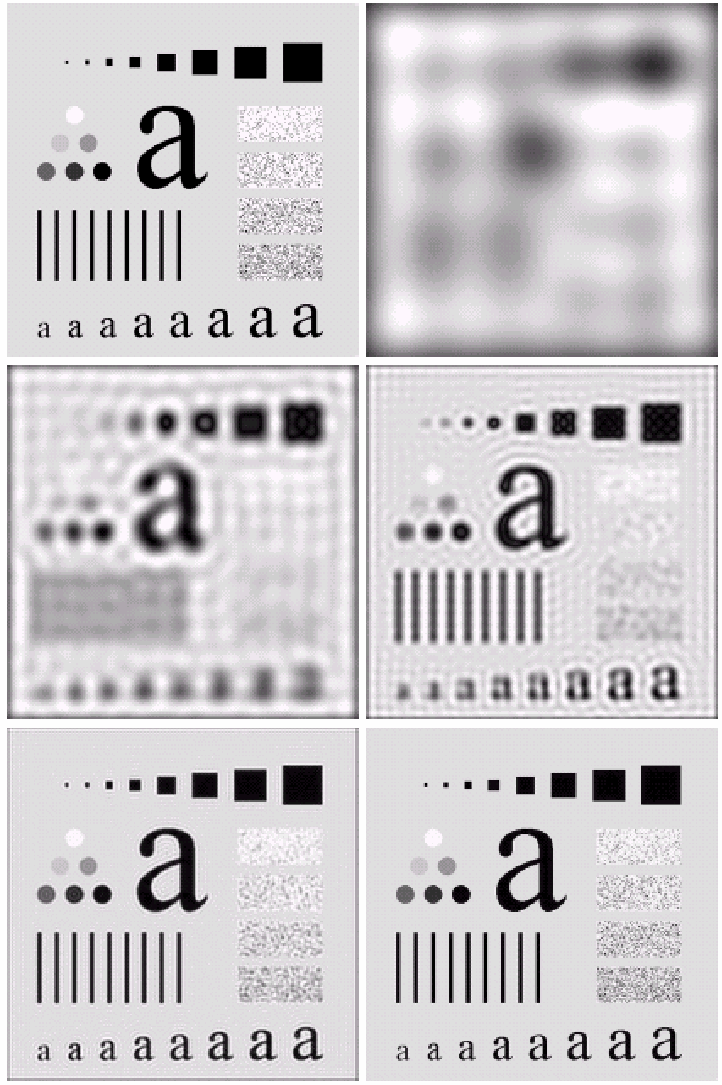
a b c

(a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.



a b

(a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.



a	b
c	d
e	f

(a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

Gaussian low pass filtering

a b c

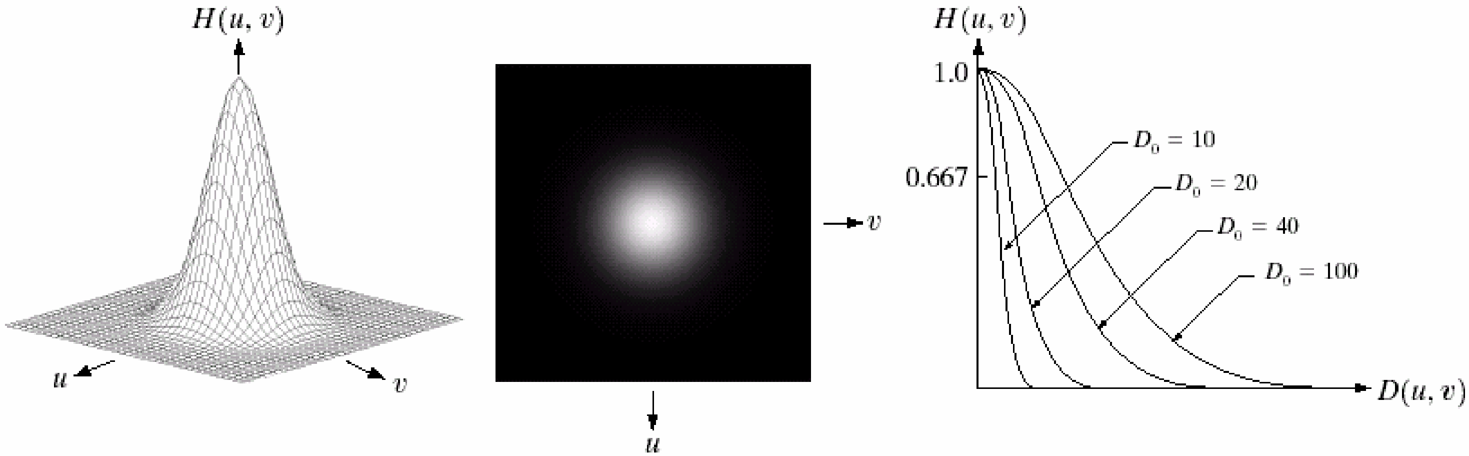
(a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)

$$H(u,v) = Ae^{-D^2(u,v)/2\sigma^2}$$

$$Let\ \sigma = D_0$$

$$H(u,v) = Ae^{-D^2(u,v)/2D_0^2}$$

GLPF



a b c

(a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

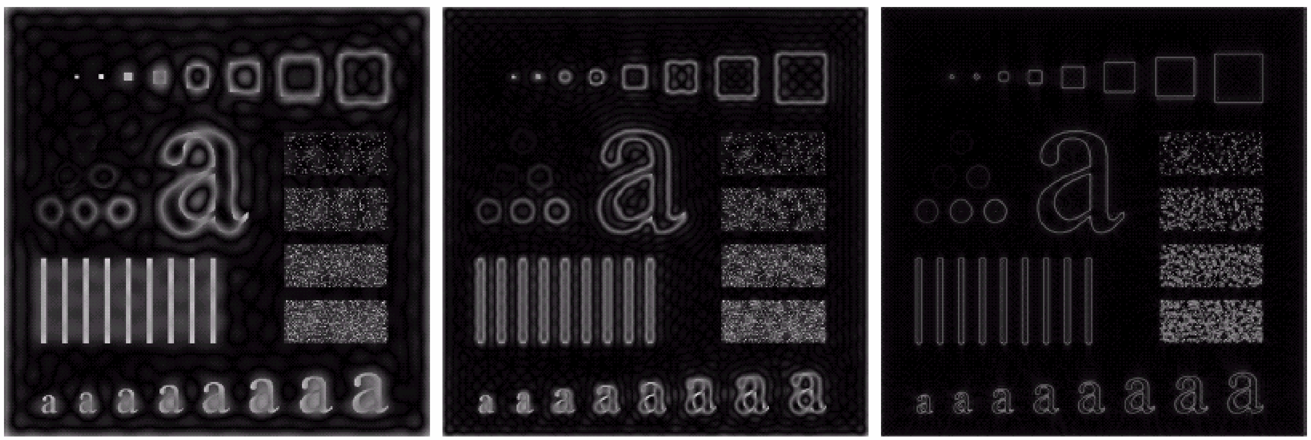
Sharpening frequency domain filters Ideal high pass filter

- Edges and abrupt changes in gray levels are associated with high-frequency components
- Perform reverse operation of IDLPF

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

$$H(u,v) = \begin{cases} 0 & \text{if } \mathrm{D(u,v)} \le \mathrm{D}_0 \\ 1 & \text{if } \mathrm{D(u,v)} > \mathrm{D}_0 \end{cases}$$

IDHPF

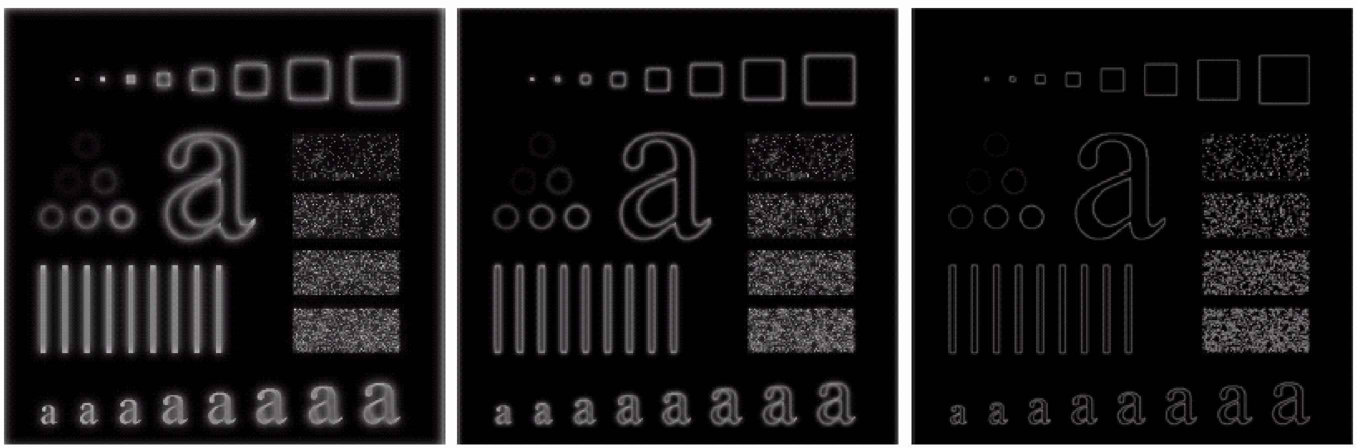


a b c

Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

Gaussian high pass filter (GHPF)

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$



a b c

Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

Laplacian in the frequency domain

$$\Im\left[\frac{d^n f(x)}{dx^n}\right] = (ju)^n F(u)$$

$$\Im\left[\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}\right] = (ju)^2 F(u,v) + (jv)^2 F(u,v)$$

$$= -(u^2 + v^2)F(u,v)$$

$$\Im\left[\nabla^2 f(x,y)\right] = - = -(u^2 + v^2)F(u,v)F(u,v)$$

Laplacian can be implemented using the filter

$$H(u,v) = -(u^2 + v^2)F(u,v)$$

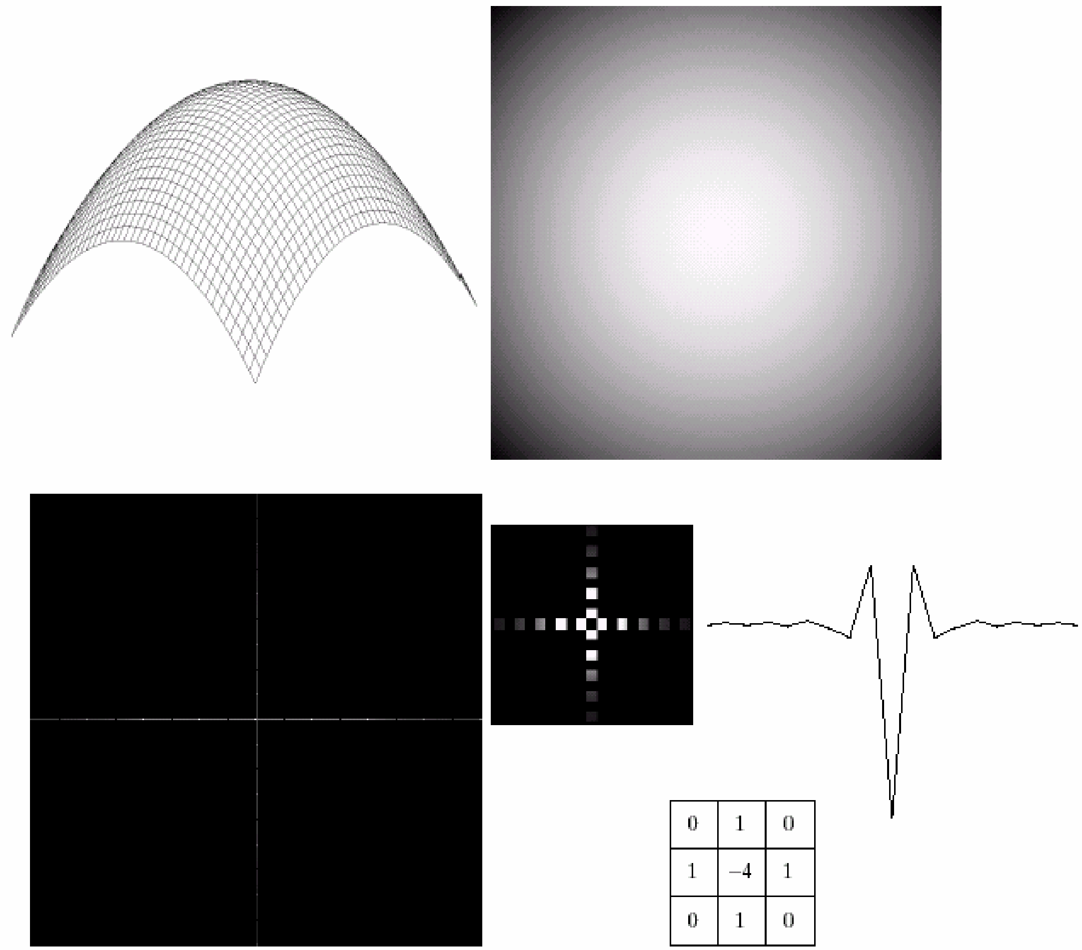
Laplacian FT pair

- Origin of F(u,v) is centered using f(x,y)(-1)x+y, before taking the transform.Center of filter function is shifted:
- $H(u,v)= -[(u-M/2)^2+(v-N/2)^2]^{1/2}$
- Laplacian filtered image in the spatial domain is obtained by computed the IFT of H(u,v)F(u,v)

$$\nabla^2 f(x,y) = \Im^{-1}\left\{-[(\mathrm{u}-\mathrm{M}/2)^2+(\mathrm{v}-\mathrm{N}/2)^2]^{1/2} F(u,v)\right\}$$

- Computing Laplacian in spatial domain and computing FT of the result is equivalent to multiplying F(u,v) by H(u,v)

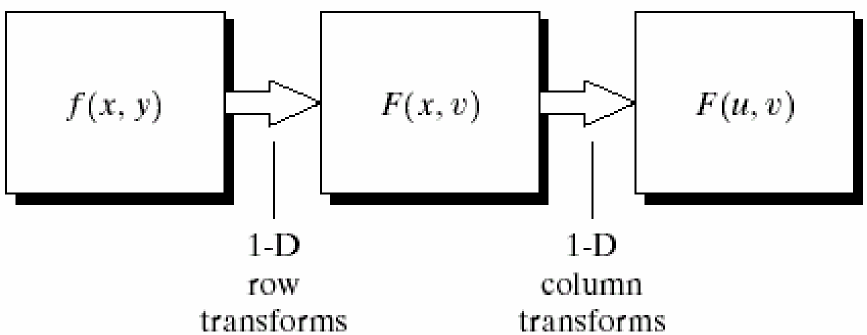
$$\nabla^2 f(x,y) \Leftrightarrow \left\{-[(\mathrm{u}-\mathrm{M}/2)^2+(\mathrm{v}-\mathrm{N}/2)^2]^{1/2} F(u,v)\right\}$$



a b
c d e
f

a) 3-D plot of Laplacian in the frequency domain. (b) Image representation of (a). (c) Laplacian in the spatial domain obtained from the inverse DFT of (b). (d) Zoomed section of the origin of (c). (e) Gray-level profile through the center of (d). (f) Laplacian mask used in Section 3.7.

FT is separable



Computation of the 2-D Fourier transform as a series of 1-D transforms.

Summary of some important properties of the 2-D Fourier transform.

Property	Expression(s)
Fourier transform	$F(u,v) = \dfrac{1}{MN}\displaystyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
Inverse Fourier transform	$f(x,y) = \displaystyle\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
Polar representation	$F(u,v) = \lvert F(u,v)\rvert e^{-j\phi(u,v)}$
Spectrum	$\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$, $R = \text{Real}(F)$ and $I = \text{Imag}(F)$
Phase angle	$\phi(u,v) = \tan^{-1}\left[\dfrac{I(u,v)}{R(u,v)}\right]$
Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
Average value	$\bar{f}(x,y) = F(0,0) = \dfrac{1}{MN}\displaystyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$
Translation	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$

Conjugate symmetry	$F(u, v) = F^*(-u, -v)$ $\lvert F(u, v)\rvert = \lvert F(-u, -v)\rvert$
Differentiation	$\dfrac{\partial^n f(x, y)}{\partial x^n} \Leftrightarrow (ju)^n F(u, v)$ $(-jx)^n f(x, y) \Leftrightarrow \dfrac{\partial^n F(u, v)}{\partial u^n}$
Laplacian	$\nabla^2 f(x, y) \Leftrightarrow -(u^2 + v^2)F(u, v)$
Distributivity	$\Im[f_1(x, y) + f_2(x, y)] = \Im[f_1(x, y)] + \Im[f_2(x, y)]$ $\Im[f_1(x, y) \cdot f_2(x, y)] \neq \Im[f_1(x, y)] \cdot \Im[f_2(x, y)]$
Scaling	$af(x, y) \Leftrightarrow aF(u, v), f(ax, by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert} F(u/a, v/b)$
Rotation	$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ $f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$
Periodicity	$F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N)$ $f(x, y) = f(x + M, y) = f(x, y + N) = f(x + M, y + N)$
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.

Property	Expression(s)
Computation of the inverse Fourier transform using a forward transform algorithm	$\frac{1}{MN} f^*(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting the function $F^*(u, v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x, y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution†	$f(x, y) * h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$
Correlation†	$f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n)$
Convolution theorem†	$f(x, y) * h(x, y) \Leftrightarrow F(u, v) H(u, v)$; $f(x, y) h(x, y) \Leftrightarrow F(u, v) * H(u, v)$
Correlation theorem†	$f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v) H(u, v)$; $f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \circ H(u, v)$

Some useful FT pairs:

Impulse	$\delta(x, y) \Leftrightarrow 1$
Gaussian	$A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$
Rectangle	$\text{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow \dfrac{1}{2}\left[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)\right]$
Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow j\dfrac{1}{2}\left[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)\right]$

† Assumes that functions have been extended by zero padding.