Image Processing

Filtering in the frequency domain
Chapter 4

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Overview

• Sampling and the Fourier transform of sampled functions
• DFT of one variable
• Extension to functions of two variables
• Filtering in frequency domain
• Image smoothing using frequency domain filters
• Selective filtering
• Implementation
FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model the sampling process.
(c) Sampled function formed as the product of (a) and (b).
(d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)
FIGURE 4.6
(a) Fourier transform of a band-limited function.
(b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.
FIGURE 4.7
(a) Transform of a band-limited function.
(b) Transform resulting from critically sampling the same function.
FIGURE 4.8
Extracting one period of the transform of a band-limited function using an ideal lowpass filter.
**FIGURE 4.9** (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.
Function reconstruction from sampled data

**FIGURE 4.10** Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. $\Delta T$ is the separation between samples.
2D function and a section of its spectrum

**FIGURE 4.13** (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the $t$-axis, so the spectrum is more “contracted” along the $\mu$-axis. Compare with Fig. 4.4.
2D impulse train
2D sampling theorem

• A continuous band-limited function $f(t,z)$ can be recovered with no error from a set of its samples if it is represented by samples acquired at rates greater than twice the highest frequency content of the function in both the directions
Aliasing in images

**Figure 4.15**
Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.
Extension from 1D aliasing

• A 2D function is band limited only if it extends infinitely in both coordinate directions
• Aliasing is always present in digital images
• Spatial aliasing is due to under-sampling
• Temporal aliasing is related to time intervals between images in a sequence of images
• Spatial aliasing-introduction of artifacts such as jaggedness in line features, spurious highlights and appearance of frequency patterns not present in image
example

• 96x96 pixels sampling of an imaging system: for digitizing a checkerboard pattern, it can resolve patterns that are up to 96x96 squares, size of each square is 1x1 pixels.
FIGURE 4.16 Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a “normal” image.
• Effect of aliasing can be reduced by defocusing the scene to be digitized so that high frequencies are attenuated

• Anti-aliasing filtering has to be done at the front-end before the image is sampled

• Anti-aliasing-blurring a digital image to reduce additional aliasing artifacts caused by resampling
Image interpolation and resampling

• Perfect reconstruction of a band-limited image function from a set of its samples requires 2D convolution in the spatial domain with a sinc function
• Requires infinite summations –forces to look for approximations
• 2d interpolation is used in image resizing (zooming and shrinking)
• Zooming-oversampling, shrinking-undersampling
• Applied to digital images
Image Interpolation

- Nearest-neighbor
- Bilinear interpolation: \( v(x, y) = ax + by + cxy + d \)
- Bicubic interpolation

\[
v(x, y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x^i y^j
\]
Zooming by pixel replication

• Double the size of an image, duplicate each column
• Then duplicate each row
• Shrinking-row column deletion
• Reduce to half size-delete every other row and column
• To reduce aliasing, blur the image slightly before shrinking it
• Another option, supersample to bigger size and reduce resample its size by row and column deletion-yields sharper results
**FIGURE 4.17** Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a $3 \times 3$ averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)
• Images with strong edge content, show block-like components called jaggies-effect of aliasing.

**FIGURE 4.18** Illustration of jaggies. (a) A 1024 × 1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5 × 5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)
In (b) effect of jags - blocky edges reduced by using bilinear interpolation better than by pixel replication (a)

**FIGURE 4.19** Image zooming. (a) A 1024 × 1024 digital image generated by pixel replication from a 256 × 256 image extracted from the middle of Fig. 4.18(a). (b) Image generated using bi-linear interpolation, showing a significant reduction in jaggies.
Moire patterns—another artifact

• Results from sampling scenes with periodic or nearly periodic components
• Seen in interference between TV raster lines and striped materials.
• Moire effect using ink drawings that have not been digitized
• Superimposing one pattern on another creates a beat pattern that has frequencies not present in either of the original patterns
FIGURE 4.20
Examples of the moiré effect. These are ink drawings, not digitized patterns. Superimposing one pattern on the other is equivalent mathematically to multiplying the patterns.
FIGURE 4.21
A newspaper image of size 246 × 168 pixels sampled at 75 dpi showing a moiré pattern. The moiré pattern in this image is the interference pattern created between the ±45° orientation of the halftone dots and the north–south orientation of the sampling grid used to digitize the image.
FIGURE 4.22
A newspaper image and an enlargement showing how halftone dots are arranged to render shades of gray.
Fourier spectrum and phase angle

FIGURE 4.24
(a) Image, (b) Spectrum showing bright spots in the four corners, (c) Centered spectrum, (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.
• The size of Fourier spectrum and phase angle is MXN for discrete variables u=0,1,2,..,M-1 and v=0,1,2,...,N-1
• The Fourier transform of a real function is conjugate symmetric which implies that the spectrum has even symmetry about the origin
• The phase angle exhibits odd symmetry
• Fig. 4.24: Spectrum values are scaled [0,255]
• Area around origin has highest value, 4 corners have high value due to periodicity property
• Centered spectrum (-1)^x+y
• Increase dynamic range of intensities by log transformation: (1+log|F(u,v)|)
FIGURE 4.25
(a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum.
(c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).
• Components of spectrum of DFT determines the amplitues of the sinusoids that combine to form the resulting image.
• At any given frequency in the DFT of an image, a large amplitude implies a greater prominent of a sinusoid of that frequency in the image.
• A small amplitude implies less of that sinusoid present in the image.
• Phase is a measure of displacement of the various sinusoids with respect to the origin.
• 2DDFT is an array whose components determine the intensities in the image.
• Phase is an array of angles with information about where discernable objects are located.
FIGURE 4.26 Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).
FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.
• 4.27(a) original face image
• 4.27(b) phase angle of DFT of (a) has no detail
• 4.27(c) inv DFT using only phase information (|F(u,v)| = 1): intensity information is lost key shape characteristics are present
• 4.27(d) : using only spectrum, setting exponent to 1 or phase angle to 0. It contains only intensity information, dc term is dominant
Implementation - Properties of the 2-D Fourier Transform

**Distributivity and scaling**

FT is distributive over addition, but not over multiplication.

\[ \mathcal{F}\{f_1(x, y) + f_2(x, y)\} = \mathcal{F}\{f_1(x, y)\} + \mathcal{F}\{f_2(x, y)\} \]

\[ \mathcal{F}\{f_1(x, y) \cdot f_2(x, y)\} \neq \mathcal{F}\{f_1(x, y)\} \cdot \mathcal{F}\{f_2(x, y)\} \]

And for Inverse Fourier Transform:

\[ af(x, y) \iff aF(u, v) \]

\[ f(ax, by) \iff \frac{1}{|ab|} F(u/a, v/b) \]

**Rotation**

Introducing polar coordinates

\[ x = r \cos \theta \quad y = r \sin \theta \quad u = w \cos \varphi \quad v = w \sin \varphi \]

\[ f(x, y) \Rightarrow f(r, \theta) \quad F(u, v) \Rightarrow F(w, \varphi) \]

Substituting into the definition of the Fourier transform:

\[ f(r, \theta + \theta_0) \iff F(w, \varphi + \theta_0) \]
Implementation - Properties of the 2-D Fourier Transform

**Periodicity and conjugate symmetry**

Periodicity for FT
\[ F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N) \]

Periodicity for Inverse transform
\[ f(x, y) = f(x + M, y) = f(x, y + N) = f(x + M, y + N) \]

Conjugate symmetry
\[ |F(u, v)| = |F(-u, -v)| \]
Implementation - Properties of the 2-D Fourier Transform

**Some symmetry properties**

<table>
<thead>
<tr>
<th>Spatial Domain</th>
<th>Frequency Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) $f(x, y)$ real</td>
<td>$F^*(u, v) = F(-u, -v)$</td>
</tr>
<tr>
<td>2) $f(x, y)$ imaginary</td>
<td>$F^*(-u, -v) = -F(u, v)$</td>
</tr>
<tr>
<td>3) $f(x, y)$ real</td>
<td>$R(u, v)$ even; $I(u, v)$ odd</td>
</tr>
<tr>
<td>4) $f(x, y)$ imaginary</td>
<td>$R(u, v)$ odd; $I(u, v)$ even</td>
</tr>
<tr>
<td>5) $f(-x, -y)$ real</td>
<td>$F^*(u, v)$ complex</td>
</tr>
<tr>
<td>6) $f(-x, -y)$ complex</td>
<td>$F(-u, -v)$ complex</td>
</tr>
<tr>
<td>7) $f^*(x, y)$ complex</td>
<td>$F^*(-u - v)$ complex</td>
</tr>
<tr>
<td>8) $f(x, y)$ real and even</td>
<td>$F(u, v)$ real and even</td>
</tr>
<tr>
<td>9) $f(x, y)$ real and odd</td>
<td>$F(u, v)$ imaginary and odd</td>
</tr>
<tr>
<td>10) $f(x, y)$ imaginary and even</td>
<td>$F(u, v)$ imaginary and even</td>
</tr>
<tr>
<td>11) $f(x, y)$ imaginary and odd</td>
<td>$F(u, v)$ real and odd</td>
</tr>
<tr>
<td>12) $f(x, y)$ complex and even</td>
<td>$F(u, v)$ complex and even</td>
</tr>
<tr>
<td>13) $f(x, y)$ complex and odd</td>
<td>$F(u, v)$ complex and odd</td>
</tr>
</tbody>
</table>

*Recall that $x, y, u,$ and $v$ are *discrete* (integer) variables, with $x$ and $u$ in the range $[0, M - 1]$, and $y$ and $v$ in the range $[0, N - 1]$. To say that a complex function is *even* means that its real and *imaginary* parts are even, and similarly for an odd complex function.
Implementation - Properties of the 2-D Fourier Transform

Separability

We can express the FT as:

\[
F(u, v) = \frac{1}{M} \sum_{x=0}^{M-1} e^{-j2\pi ux/M} \frac{1}{N} \sum_{y=0}^{N-1} f(x, y)e^{-j2\pi vy/N}
\]

\[
= \frac{1}{M} \sum_{x=0}^{M-1} F(x, v)e^{-j2\pi ux/M}
\]

where

\[
F(x, v) = \frac{1}{N} \sum_{y=0}^{N-1} f(x, y)e^{-j2\pi vy/N}
\]

We can compute the 2-D transform by first computing a 1-D transform along each row of the input image, and the computing a 1-D transform along each column of this intermediate result: columns first, followed by rows.
Implementation

More on Periodicity: The Need for Padding

The significance of periodicity in a convolution:

\[ f(x) * h(x) = \frac{1}{M} \sum_{m=0}^{M-1} f(m) h(x - m) \]
• DFT is faster with arrays of even size. If the two arrays are of same size, P and Q are selected twice the array size

• Frequency leakage-caused by high-freq. components of the sinc function

• Leakage produces a blocky effect, can be reduced by multiplying the sampled function by another function that tapers smoothly to near zero both ends to dampen the sharp transitions – windowing

• 2D Gaussian is a good window function
Implementation

The convolution and correlation Theorems

The discrete convolution of two functions \( f(x,y) \) and \( h(x,y) \) of size \( M \times N \) is defined by the expression:

\[
f(x,y) \ast h(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)
\]

Relationship between the two functions and their Fourier transforms:

\[
f(x,y) \ast h(x,y) \Leftrightarrow F(u,v)H(u,v) \quad f(x,y)h(x,y) \Leftrightarrow F(u,v) \ast H(u,v)
\]

The correlation of two functions:

\[
f(x,y) \ast h(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^\ast(m,n)h(x+m,y+n)
\]

We can compress the correlation theorem in the next two results:

\[
f(x,y) \ast h(x,y) = F \ast (u,v)H(u,v) \quad f^\ast(x,y)h(x,y) \Leftrightarrow F(u,v) \ast H(u,v)
\]

The autocorrelation Theorem:

\[
f(x,y) \ast h(x,y) \Leftrightarrow |F(u,v)|^2
\]

Similarly

\[
|f(x,y)|^2 \Leftrightarrow F(u,v) \ast F(u,v)
\]
**Summary of definitions and corresponding expression of DFT (I)**

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) Discrete Fourier transform (DFT) of ( f(x, y) )</td>
<td>( F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)} )</td>
</tr>
<tr>
<td>2) Inverse discrete Fourier transform (IDFT) of ( F(u, v) )</td>
<td>( f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)} )</td>
</tr>
<tr>
<td>3) Polar representation</td>
<td>( F(u, v) =</td>
</tr>
</tbody>
</table>
| 4) Spectrum | \( |F(u, v)| = \left[ R^2(u, v) + I^2(u, v) \right]^{1/2} \)  
\( R = \text{Real}(F), \quad I = \text{Imag}(F) \) |
| 5) Phase angle | \( \phi(u, v) = \tan^{-1}\left( \frac{I(u, v)}{R(u, v)} \right) \) |
| 6) Power spectrum | \( P(u, v) = |F(u, v)|^2 \) |
| 7) Average value | \( \overline{f}(x, y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = \frac{1}{MN} F(0, 0) \) |

(Continued)
### Summary of definitions and corresponding expression of DFT (II)

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression(s)</th>
</tr>
</thead>
</table>
| 8) Periodicity ($k_1$ and $k_2$ are integers) | $F(u, v) = F(u + k_1M, v) = F(u, v + k_2N)$  
$ = F(u + k_1M, v + k_2N)$  
$f(x, y) = f(x + k_1M, y) = f(x, y + k_2N)$  
$ = f(x + k_1M, y + k_2N)$  
$M-1 \leq k_1 \leq N-1$  
$N-1 \leq k_2 \leq M-1$                                                                 |**TABLE 4.2**  
(Continued)                                                                                                                                 |
| 9) Convolution                            | $f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x - m, y - n)$                                                                                                                                 |
| 10) Correlation                           | $f(x, y) \circ h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n)h(x + m, y + n)$                                                                                                                                 |
| 11) Separability                          | The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.1. |                                                                                                                                               |
| 12) Obtaining the inverse Fourier transform using a forward transform algorithm. | $MNf^*(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v)e^{-j2\pi vx/M}e^{-j2\pi ux/N}$  
This equation indicates that inputting $F^*(u, v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x, y)$. Taking the complex conjugate and dividing by $MN$ gives the desired inverse. See Section 4.11.2. |
## Summary of DTF pairs (I)

<table>
<thead>
<tr>
<th>Name</th>
<th>DFT Pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) Symmetry properties</td>
<td>See Table 4.1</td>
</tr>
<tr>
<td>2) Linearity</td>
<td>( af_1(x, y) + bf_2(x, y) \Leftrightarrow aF_1(u, v) + bF_2(u, v) )</td>
</tr>
<tr>
<td>3) Translation (general)</td>
<td>( f(x, y)e^{j2\pi(ux/M + vy/N)} \Leftrightarrow F(u - u_0, v - v_0) )</td>
</tr>
<tr>
<td></td>
<td>( f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(ux_0/M + vy_0/N)} )</td>
</tr>
<tr>
<td>4) Translation to center of the frequency</td>
<td>( f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2) )</td>
</tr>
<tr>
<td>rectangle, ((M/2, N/2))</td>
<td>( f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v} )</td>
</tr>
<tr>
<td>5) Rotation</td>
<td>( f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0) )</td>
</tr>
<tr>
<td></td>
<td>( x = r \cos \theta \quad y = r \sin \theta \quad u = \omega \cos \varphi \quad v = \omega \sin \varphi )</td>
</tr>
<tr>
<td>6) Convolution theorem†</td>
<td>( f(x, y) \ast h(x, y) \Leftrightarrow F(u, v)H(u, v) )</td>
</tr>
<tr>
<td></td>
<td>( f(x, y)h(x, y) \Leftrightarrow F(u, v) \ast H(u, v) )</td>
</tr>
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</table>

*Continued*
Summary of DFT pairs (II)

<table>
<thead>
<tr>
<th>Name</th>
<th>DFT Pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>7) Correlation theorem</td>
<td>( f(x, y) \ast h(x, y) \Leftrightarrow F^*(u, v) H(u, v) )</td>
</tr>
<tr>
<td></td>
<td>( f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \ast H(u, v) )</td>
</tr>
<tr>
<td>8) Discrete unit impulse</td>
<td>( \delta(x, y) \Leftrightarrow 1 )</td>
</tr>
<tr>
<td>9) Rectangle</td>
<td>( \text{rect}[a, b] \Leftrightarrow ab \frac{\sin(\pi u a)}{\pi u a} \frac{\sin(\pi v b)}{\pi v b} e^{-j\pi(u a + v b)} )</td>
</tr>
<tr>
<td>10) Sine</td>
<td>( \sin(\pi u x + \pi v y) \Leftrightarrow \frac{1}{2} \left[ \delta(u + Mu_0, v + Nv_0) - \delta(u - Mu_0, v - Nv_0) \right] )</td>
</tr>
<tr>
<td>11) Cosine</td>
<td>( \cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow \frac{1}{2} \left[ \delta(u + Mu_0, v + Nv_0) + \delta(u - Mu_0, v - Nv_0) \right] )</td>
</tr>
</tbody>
</table>

The following Fourier transform pairs are derivable only for continuous variables, denoted as before by \( t \) and \( z \) for spatial variables and by \( \mu \) and \( \nu \) for frequency variables. These results can be used for DFT work by sampling the continuous forms.

12) Differentiation
   (The expressions on the right assume that \( f(\pm \infty, \pm \infty) \equiv 0 \))
   \[
   \frac{\partial^n}{\partial t^n} f(t, z) \Leftrightarrow (j2\pi \mu)^n (j2\pi \nu)^n F(\mu, \nu)
   \]
   \[
   \frac{\partial^n}{\partial z^n} f(t, z) \Leftrightarrow (j2\pi \mu)^n (j2\pi \nu)^n F(\mu, \nu)
   \]

13) Gaussian
   \[
   A \exp(-\pi^2 u^2 - \pi^2 v^2) \Leftrightarrow \frac{A}{\sqrt{\pi^2 u^2 + \pi^2 v^2}} \exp(-\pi^2 u^2 - \pi^2 v^2)
   \]

\(^{1}\text{Assumes that the functions have been extended by zero padding. Convolution and correlation are associative, commutative, and distributive.}\)
Basics of filtering in the frequency domain

- Each term of $F(u,v)$ contains all values of $f(x,y)$ modified by values of exp. Terms
- Slowest varying freq. component ($u=v=0$) is proportional to the average intensity of an image
- Low freq. correspond to slowly varying intensity components of an image
- High frequencies correspond to faster intensity changes in image (edges, etc)
FIGURE 4.29 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)
• Fig 4.29

• Fourier spectrum shows prominent components along the 45deg directions that correspond to the edges

• Vertical component off-axis slightly to left- due to oxide protrusions
Frequency domain filtering fundamentals

FIGURE 4.30
Result of filtering the image in Fig. 4.29(a) by setting to 0 the term $F(M/2, N/2)$ in the Fourier transform.
Adding a constant ‘a’ to the filter does not affect sharpening, but does prevent elimination of the dc term and thus prevents tonality.
FIGURE 4.32  (a) A simple image.  (b) Result of blurring with a Gaussian lowpass filter without padding.  (c) Result of lowpass filtering with padding.  Compare the light area of the vertical edges in (b) and (c).
**FIGURE 4.33** 2-D image periodicity inherent in using the DFT. (a) Periodicity without image padding. (b) Periodicity after padding with 0s (black). The dashed areas in the center correspond to the image in Fig. 4.32(a). (The thin white lines in both images are superimposed for clarity; they are not part of the data.)
Filter is real and has even symmetry, IDFT will be real and symmetric

**FIGURE 4.34**
(a) Original filter specified in the (centered) frequency domain.
(b) Spatial representation obtained by computing the IDFT of (a).
(c) Result of padding (b) to twice its length (note the discontinuities).
(d) Corresponding filter in the frequency domain obtained by computing the DFT of (c). Note the ringing caused by the discontinuities in (c). (The curves appear continuous because the points were joined to simplify visual analysis.)
• FT of a box function is a sinc function with freq. components extending to infinity
• Cannot use an ideal freq. domain filter (Fig. 4.34a) (due to ringing effect) and simultaneously use zero padding to avoid wraparound error
• Objective: Work with specified filter shapes in the frequency domain (including ideal filters)
• Zero-pad image and create filters in freq. domain to be same size as padded image
• Pad images to size PxQ and construct filters of the same dimensions (avoids ringing effect)
• \( F(u,v) = R(u,v) + jI(u,v) \)
• \( G(x,y) = F^{-1}[H(u,v)R(u,v) + jH(u,v)I(u,v)] \)
\[ F(u,v) = R(u,v) + jI(u,v) \]

Filters that affect the real and imaginary parts equally, and thus have no effect on the phase, are called Zero-phase-shift filters.

Multiplying \[ F(u,v) = |F(u,v)| e^{j\phi(u,v)} \] by 0.5 and 0.25
Correspondence between filtering in spatial and frequency domain

- $F(x,y) = \delta(x,y)$
- $F(u,v) = 1$
- Given a spatial filter obtain its freq. domain representation by taking forward FT of the spatial filter
- $h(x,y) \leftrightarrow H(u,v)$
- $h(x,y)$ is impulse response of $H(u,v)$ are FIR filters
- Spatial convolution - convolving periodic functions (circular convolution)
• In practise, we prefer to implement convolution filtering with small filter masks because of speed
• Specify a filter in freq. domain, compute its IDFT, use the full-size spatial filter as a guide for constructing smaller spatial filter masks
• The converse, given a small spatial filter-obtain its full-size freq domain representation
• This is useful for analyzing behavior of small spatial filters in freq. domain.
Gaussian filters

• Illustrate with these filters, because both forward and inverse FT of a Gaussian function are real Gaussian functions
• The freq. domain filter has all positive values, hence the lowpass spatial domain filter should be a mask with all positive coefficients
• Narrower the filter, it will attenuate more the low frequencies, increasing blurring
• We can construct high pass filters as difference of Gaussians
FIGURE 4.36
(a) An $M \times N$ image, $f$.
(b) Padded image, $f_p$ of size $P \times Q$.
(c) Result of multiplying $f_p$ by $(-1)^{x+y}$.
(d) Spectrum of $F_p$.
(e) Centered Gaussian lowpass filter, $H$, of size $P \times Q$.
(f) Spectrum of the product $HF_p$.
(g) $g_p$, the product of $(-1)^{x+y}$ and the real part of the IDFT of $HF_p$.
(h) Final result, $g$, obtained by cropping the first $M$ rows and $N$ columns of $g_p$. 
FIGURE 4.37
(a) A 1-D Gaussian lowpass filter in the frequency domain. (b) Spatial lowpass filter corresponding to (a). (c) Gaussian highpass filter in the frequency domain. (d) Spatial highpass filter corresponding to (c). The small 2-D masks shown are spatial filters we used in Chapter 3.
FIGURE 4.38
(a) Image of a building, and
(b) its spectrum.
Figure 4.39
(a) A spatial mask and perspective plot of its corresponding frequency domain filter. (b) Filter shown as an image. (c) Result of filtering Fig. 4.38(a) in the frequency domain with the filter in (b). (d) Result of filtering the same image with the spatial filter in (a). The results are identical.
Image smoothing using frequency domain filters

• Ideal lowpass filters
FIGURE 4.40 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.
FIGURE 4.41 (a) Test pattern of size $688 \times 688$ pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2\% of the padded image power, respectively.
FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using HPFs with cutoff frequencies set at radii values 10, 30, 60, 180, and 400, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.
**FIGURE 4.43**
(a) Representation in the spatial domain of an ILPF of radius 5 and size 1000 × 1000.
(b) Intensity profile of a horizontal line passing through the center of the image.
FIGURE 4.44 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.
FIGURE 4.45 (a) Original image. (b)-(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.
FIGURE 4.46 (a)-(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is $1000 \times 1000$ and the cutoff frequency is $\lambda$). Observe how ringing increases as a function of filter order.
FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of $D_0$. 
### TABLE 4.4

Lowpass filters. $D_0$ is the cutoff frequency and $n$ is the order of the Butterworth filter.

<table>
<thead>
<tr>
<th></th>
<th>Ideal</th>
<th>Butterworth</th>
<th>Gaussian</th>
</tr>
</thead>
</table>
| $H(u, v)$ | $\begin{cases} 
1 & \text{if } D(u, v) \leq D_0 \\
0 & \text{if } D(u, v) > D_0 
\end{cases}$ | $H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}}$ | $H(u, v) = e^{-D^2(u, v)/2D_0^2}$ |
Figure 4.48 (a) Original image. (b)-(f) Results of filtering using GLPFs with cutoff frequencies at the rates shown in Fig. 4.41. Compare with Figs. 4.42 and 4.45.
Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.

**Figure 4.49**

(a) Sample text of low resolution (note broken characters in magnified view).
(b) Result of filtering with a GLPF (broken character segments were joined).
FIGURE 4.50 (a) Original image (784 × 732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).
FIGURE 4.51  (a) Image showing prominent horizontal scan lines. (b) Result of filtering using a GLPF with \( D_0 = 50 \). (c) Result of using a GLPF with \( D_0 = 20 \). (Original image courtesy of NOAA.)
FIGURE 4.52  Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.
FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.
FIGURE 4.54 Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60, \text{ and } 160$. 
FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60, \text{and } 160$, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.
**FIGURE 4.56** Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30, 60,$ and $160$, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.
**TABLE 4.5**

Highpass filters. $D_0$ is the cutoff frequency and $n$ is the order of the Butterworth filter.

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Butterworth</th>
<th>Gaussian</th>
</tr>
</thead>
</table>
| $H(u, v) = \begin{cases} 
1 & \text{if } D(u, v) \leq D_0 \\
0 & \text{if } D(u, v) > D_0
\end{cases}$ | $H(u, v) = \frac{1}{1 + \left[ \frac{D_0}{D(u, v)} \right]^{2n}}$ | $H(u, v) = 1 - e^{-D^2(u,v)/2D_0^2}$ |
FIGURE 4.57 (a) Thumb print. (b) Result of highpass filtering (a). (c) Result of thresholding (b). (Original image courtesy of the U.S. National Institute of Standards and Technology.)
FIGURE 4.58
(a) Original, blurry image.
(b) Image enhanced using the Laplacian in the frequency domain. Compare with Fig. 3.38(e).
Figure 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)
Homomorphic Filtering

An image $f(x,y)$ can be expressed as a product of illumination and reflectance:

$$f(x,y) = i(x,y)r(x,y)$$

The desired enhanced image can be expressed denoted by:

$$g(x,y) = i_0(x,y)r_0(x,y)$$

$i_0(x,y)$ and $r_0(x,y)$ are the illumination and reflectance components of the output image. We can summarized the concepts in the next sequence of steps.

![Sequence of steps in homomorphic filtering](image)

Where $H(u,v)$ is the homomorphic filter function that affects the low and high frequency components of the Fourier transform as we can see in the next plot:

![Radial cross section of a circularly symmetric homomorphic filter function](image)
Homomorphic Filtering – Example

Using a slightly modified form of the Gaussian highpass filter:

\[ H(u, v) = (\gamma_H - \gamma_L) \left[ 1 - e^{-c(D^2(u,v)/D_0^2)} \right] + \gamma_L \]

FIGURE 4.62
(a) Full body PET scan. (b) Image enhanced using homomorphic filtering. (Original image courtesy of Dr. Michael E. Casey, CTI PET Systems.)
FIGURE 4.61
Radial cross section of a circularly symmetric homomorphic filter function. The vertical axis is at the center of the frequency rectangle and $D(u, v)$ is the distance from the center.
FIGURE 4.62
(a) Full body PET scan, (b) Image enhanced using homomorphic filtering. (Original image courtesy of Dr. Michael E. Casey, CTI PET Systems.)
### Table 4.6

Bandreject filters. $W$ is the width of the band, $D$ is the distance $D(u, v)$ from the center of the filter, $D_0$ is the cutoff frequency, and $n$ is the order of the Butterworth filter. We show $D$ instead of $D(u, v)$ to simplify the notation in the table.

<table>
<thead>
<tr>
<th></th>
<th>Ideal</th>
<th>Butterworth</th>
<th>Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(u, v)$</td>
<td>$\begin{cases} 0 &amp; \text{if } D_0 - \frac{W}{2} \leq D \leq D_0 + \frac{W}{2} \ 1 &amp; \text{otherwise} \end{cases}$</td>
<td>$H(u, v) = \frac{1}{1 + \left[ \frac{D W}{D^2 - D_0^2} \right]^{2n}}$</td>
<td>$H(u, v) = 1 - e^{\left[ \frac{(u^2 + v^2)}{D^2} \right]^2}$</td>
</tr>
</tbody>
</table>


FIGURE 4.63
(a) Bandreject Gaussian filter.  
(b) Corresponding bandpass filter.  
The thin black border in (a) was added for clarity; it is not part of the data.
• Application of notch filtering is to selectively modify local regions of the DFT.
• DFT is without padding-hence done interactively
• Figure below shows a moire pattern and its spectrum
• Result of multiplying DFT by a Butterworth notch filter with Do=3 and n=4
FIGURE 4.64
(a) Sampled newspaper image showing a moiré pattern.  
(b) Spectrum.  
(c) Butterworth notch reject filter multiplied by the Fourier transform.  
(d) Filtered image.
FIGURE 4.65
(a) \(674 \times 674\) image of the Saturn rings showing nearly periodic interference.
(b) Spectrum: The bursts of energy in the vertical axis near the origin correspond to the interference pattern. (c) A vertical notch reject filter. (d) Result of filtering. The thin black border in (c) was added for clarity; it is not part of the data. (Original image courtesy of Dr. Robert A. West, NASA/JPL.)
FIGURE 4.66
(a) Result (spectrum) of applying a notch pass filter to the DFT of Fig. 4.65(a).
(b) Spatial pattern obtained by computing the IDFT of (a).
**FIGURE 4.67**
Computational advantage of the FFT over a direct implementation of the 1-D DFT. Note that the advantage increases rapidly as a function of $n$. 
Exercises

• 4.4, 4.5, 4.7, 4.11, 4.12
• 4.13, 4.15, 4.21, 4.22
• 4.27, 4.30, 4.32, 4.36
• 4.4) Consider the continuous function \( f(t) = \sin(2\pi nt) \).

• (a) What is the period of \( f(t) \)?
• (b) What is the frequency of \( f(t) \)?

4.12) Consider a checkerboard image in which each square is 1x1 mm. Assuming that the image extends infinitely in both coordinate directions, what is the minimum sampling rate (in samples/mm) required to avoid aliasing.

4.13) We know that shrinking an image can cause aliasing. Is this true also of zooming? Explain.
Image Padding Section 4.6.6

• Filtering in freq. domain (2D convolution).-pad images.