Problem One: LINEARITY
(25 points) Show that the following system IS NOT LINEAR:

\[ y[n] = T \{ x[n] \} = e^{x[n+1]+x[n-1]} \]

Solution to Problem One: A system T is linear if it satisfies the following condition or identity:

\[ T \{ ax_1[n] + bx_2[n] \} = aT \{ x_1[n] \} + bT \{ x_2[n] \} \]

Computing the right hand side (r.h.s.) of the identity, we have:

\[ aT \{ x_1[n] \} = ae^{x_1[n+1]+x_1[n-1]} = ae^{x_1[n+1]} e^{x_1[n-1]} \]
\[ bT \{ x_2[n] \} = be^{x_2[n+1]+x_2[n-1]} = be^{x_2[n+1]} e^{x_2[n-1]} \]
\[ aT \{ x_1[n] \} + bT \{ x_2[n] \} = ae^{x_1[n+1]} e^{x_1[n-1]} + be^{x_2[n+1]} e^{x_2[n-1]} \]

To compute the left hand side (l.h.s.) of the identity, we proceed as follows:
Use an intermediate step:

\[ x_3[n] = ax_1[n] + bx_2[n] \]

Compute, by induction, the output of the system \( T \) as follows:

\[ T \{ x_3[n] \} = e^{x_3[n+1]+x_3[n-1]} = e^{x_3[n+1]} e^{x_3[n-1]} \]

Next, we substitute for \( x_3[n] = ax_1[n] + bx_2[n] \) and obtain:

Q.E.D.: "quod erat demonstrandum" ("that which was to be demonstrated")
Comparing the two sides of the condition or identity, it is noticed that they are not equal and, hence, the system IS NOT LINEAR (Q.E.D.).

Another Solution Approach: Problem One (Concise and Elegant):
For a system \( T \) to be linear it must satisfy the homogeneity condition:

\[
T \{ ax[n] \} = a T \{ x[n] \}
\]

Computing the r.h.s. of the identity, we obtain:

\[
T \{ ax[n] \} = e^{ax[n+1]+ax[n-1]} = e^{ax[n+1]} e^{ax[n-1]}
\]

Computing the l.h.s. of the identity, we obtain:

\[
a T \{ x[n] \} = ae^{x[n+1]+x[n-1]} = ae^{x[n+1]} e^{x[n-1]}
\]

Comparing the two sides of the condition or identity, it is noticed that they are not equal and, hence, the system IS NOT LINEAR (Q.E.D.).

Q.E.D.: "quod erat demonstrandum" ("that which was to be demonstrated")
Problem Two: TIME INVARIANCE
(25 points) Determine if the system below IS TIME INVARIANT:

\[ y[n] = T \{ x[n] \} = e^{x[n+1]+x[n-1]} \]

Solution to Problem Two: A system \( T \) is time invariant or T.I. if it satisfies the following condition or identity:

\[ T \{ x[n-n_0] \} = y[n-n_0] \]

This is for the equation of the system given by \( y[n] = T \{ x[n] \} \)

We proceed by computing the r. h. s. of the condition or identity:

The equation of the system is \( y[n] = T \{ x[n] \} = e^{x[n+1]+x[n-1]} \), we then have

\[ y[(n-n_0)] = e^{x[(n-n_0)+1]+x[(n-n_0)-1]} = e^{x[n-n_0+1]}e^{x[n-n_0-1]} \]

To compute the l. h. s. of the identity, we first introduce an intermediate step:

Let \( x[n-n_0] = g[n] \). We then have

\[ T \{ g[n] \} = e^{g[n+1]+g[n-1]} \]

Finally, we substitute for \( x[n-n_0] = g[n] \) and obtain the following:

\[ T \{ x[n-n_0] \} = e^{x[(n+1)-n_0]+x[(n-1)-n_0]} = e^{x[n+1-n_0]}e^{x[n-1-n_0]} \]

Comparing the two sides of the identity, we notice that they are equal, and, hence, the system \( T \) defined above IS TIME INVARIANT.

Q.E.D.: "quod erat demonstrandum" ("that which was to be demonstrated")
Problem Three: LINEAR CONVOLUTION

(50 points) Obtain the output of the following system below:

\[ y[n] = T \{ x[n] \} = x[n] * h[n], \]

where the impulse response is given by

\[ h[n] = T \{ \delta[n] \} = \{ h[n] = (-1)^n, n \in \mathbb{Z}_8 \}, \]

and the input signal is given by

\[ x[n] = \{ x[n] = \sin(\frac{2\pi n}{4}), n \in \mathbb{Z}_4 \} \]

Solution to Problem Three: To solve this problem, we first evaluate the signals \( x[n] \) and \( h[n] \):

\[
\begin{align*}
  h[0] &= (-1)^0 = +1 \\
  h[1] &= (-1)^1 = -1 \\
  h[2] &= (-1)^2 = +1 \\
  h[3] &= (-1)^3 = -1 \\
  h[4] &= (-1)^4 = +1 \\
  h[5] &= (-1)^5 = -1 \\
  h[6] &= (-1)^6 = +1 \\
  h[7] &= (-1)^7 = -1
\end{align*}
\]

To perform the linear convolution operation we use the equation:

\[
y[n] = \sum_{k=0}^{k=3} x[k] h[n-k]; \quad n \in \mathbb{Z}_{11}
\]

Q.E.D.: "quod erat demonstrandum" ("that which was to be demonstrated")
Expanding the summation expression, we obtain


Substituting for the values of \( x[n] \), we obtain

\[ y[n] = h[n-1] - h[n-3], \quad n \in Z_{11} \]

We proceed by substituting the values of \( h[n] \) in this last equation to obtain:

\[
\begin{align*}
    y[0] &= h[0-1] - h[0-3] = 0 \\
    y[4] &= h[4-1] - h[4-3] = 0 \\
    y[7] &= h[7-1] - h[7-3] = 0 \\
    y[8] &= h[8-1] - h[8-3] = 0 \\
\end{align*}
\]

Q.E.D.: "quod erat demonstrandum" ("that which was to be demonstrated")