Discrete-Time Analytic Signal Generation

- A discrete-time analytic signal has a zero-valued spectrum for all negative frequencies
- Such a signal finds applications in single-sideband digital communication systems
- We outline a method of generating a discrete-time analytic signal $y[n]$ from a discrete-time real signal $x[n]$ and describe some of its applications

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Discrete-Time Analytic Signal Generation

- Now, the Fourier transform $X(e^{j\omega})$ of a real signal $x[n]$, if it exists, is nonzero for both positive and negative frequencies
- A signal $y[n]$ with a single-sided spectrum $Y(e^{j\omega})$ that is zero for negative frequencies must be a complex signal
- Consider the complex analytic signal

$$y[n] = x[n] + j\hat{x}[n]$$

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Discrete-Time Analytic Signal Generation

where $x[n]$ and $\hat{x}[n]$ are real

- The Fourier transform $Y(e^{j\omega})$ of $y[n]$ is given by

$$Y(e^{j\omega}) = X(e^{j\omega}) + j\hat{X}(e^{j\omega})$$

where $\hat{X}(e^{j\omega})$ is the Fourier transform of $\hat{x}[n]$

- Now, $x[n]$ and $\hat{x}[n]$ being real, their corresponding Fourier transforms are conjugate symmetric

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Discrete-Time Analytic Signal Generation

- That is, $X(e^{j\omega}) = X^*(e^{-j\omega})$ and $\hat{X}(e^{j\omega}) = \hat{X}^*(e^{-j\omega})$
- Therefore, from $Y(e^{j\omega}) = X(e^{j\omega}) + j\hat{X}(e^{j\omega})$, it follows that

$$X(e^{j\omega}) = \tfrac{1}{2}[Y(e^{j\omega}) + Y^*(e^{-j\omega})]$$

$$j\hat{X}(e^{j\omega}) = \tfrac{1}{2}[Y(e^{j\omega}) - Y^*(e^{-j\omega})]$$

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Discrete-Time Analytic Signal Generation

- Since, by assumption, $Y(e^{j\omega}) = 0$ for $-\pi \le \omega < 0$, we have

$$Y(e^{j\omega}) = \begin{cases} 2X(e^{j\omega}), & 0 \le \omega < \pi \\ 0, & -\pi \le \omega < 0 \end{cases}$$

- Thus, the analytic signal $y[n]$ can be generated by passing $x[n]$ through a linear discrete-time system, with a frequency response

$$H(e^{j\omega}) = \begin{cases} 2, & 0 \le \omega < \pi \\ 0, & -\pi \le \omega < 0 \end{cases}$$

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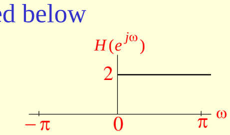
Discrete-Time Analytic Signal Generation

- The frequency response $H(e^{j\omega})$ of the system generating the analytic signal is thus as indicated below



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Discrete-Time Hilbert Transformer

- We now related the imaginary part $\hat{x}[n]$ of the analytic signal $y[n]$ to its real part $x[n]$
- From $j\hat{X}(e^{j\omega}) = \frac{1}{2}[Y(e^{j\omega}) - Y^*(e^{-j\omega})]$ we get

$$\hat{X}(e^{j\omega}) = \frac{1}{2j}[Y(e^{j\omega}) - Y^*(e^{-j\omega})]$$

- For $0 \le \omega < \pi, Y(e^{-j\omega}) = 0$, and for $-\pi \le \omega < 0$, $Y(e^{j\omega}) = 0$

Discrete-Time Hilbert Transformer

- Using the above property of $Y(e^{j\omega})$ and the relation

$$Y(e^{j\omega}) = \begin{cases} 2X(e^{j\omega}), & 0 \le \omega < \pi \\ 0, & -\pi \le \omega < 0 \end{cases}$$

it can be easily shown that

$$\hat{X}(e^{j\omega}) = \begin{cases} -jX(e^{j\omega}), & 0 \le \omega < \pi \\ jX(e^{j\omega}), & -\pi \le \omega < 0 \end{cases}$$

Discrete-Time Hilbert Transformer

- Thus, the imaginary part $\hat{x}[n]$ of the analytic signal $y[n]$ can be generated by passing its real part $x[n]$ through a linear discrete-time system with a frequency response $H_{HT}(e^{j\omega})$ given by

$$H_{HT}(e^{j\omega}) = \begin{cases} -j, & 0 \le \omega < \pi \\ j, & -\pi \le \omega < 0 \end{cases}$$

- The above linear system is usually referred to as the Hilbert transformer

Discrete-Time Hilbert Transformer

- The output $\hat{x}[n]$ of the Hilbert transformer is called the Hilbert transform of its input $x[n]$
- The basic scheme for the generation of an analytic signal $y[n] = y_{re}[n] + j\,y_{im}[n]$ from a real signal $x[n]$ is thus as indicated below:

Discrete-Time Hilbert Transformer

- Observe that $|H(e^{j\omega})| = 1$ for all frequencies and has a -90-degree phase shift for and $+90$-degree phase shift for
- As a result, an ideal Hilbert transformer is also called a 90-degree-phase-shifter
- The impulse response $h_{HT}[n]$ of an ideal Hilbert transformer is given by

$$h_{HT}[n] = \begin{cases} 0, & \text{for } n \text{ even} \\ \frac{2}{\pi n}, & \text{for } n \text{ odd} \end{cases}$$

Discrete-Time Hilbert Transformer

- Since the ideal Hilbert transformer has a two-sided infinite-length impulse response defined for $-\pi < n < \pi$, it is an unrealizable system
- Moreover, its transfer function $H_{HT}(z)$ exists only on the unit circle
- We next describe two approaches for developing a realizable approximation

Relation with Half-Band Filters

- Consider the filter with a frequency response $G(e^{j\omega})$ obtained by shifting the frequency response

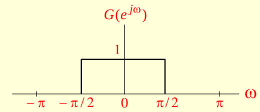
$$H(e^{j\omega}) = \begin{cases} 2, & 0 \le \omega < \pi \\ 0, & -\pi \le \omega < 0 \end{cases}$$

by $\pi/2$ radians and scaling by a factor $1/2$:

$$G(e^{j\omega}) = \tfrac{1}{2} H(e^{j(\omega+\pi/2)}) = \begin{cases} 1, & 0 < |\omega| < \pi/2 \\ 0, & -\pi/2 < |\omega| < \pi \end{cases}$$

Relation with Half-Band Filters

- A plot of $G(e^{j\omega})$ is shown below:



- Thus, $G(e^{j\omega})$ is the frequency response of a half-band lowpass filter
- $G(e^{j\omega})$ has been referred to as a complex half-band filter

Design of the Hilbert Transformer

- It follows from the relation

$$G(e^{j\omega}) = \tfrac{1}{2} H(e^{j(\omega+\pi/2)}) = \begin{cases} 1, & 0 < |\omega| < \pi/2 \\ 0, & -\pi/2 < |\omega| < \pi \end{cases}$$

that a complex half-band filter can be designed by simply shifting the frequency response of a half-band lowpass filter by $\pi/2$ radians and then scaling by a factor 2

Design of the Hilbert Transformer

- Equivalently, the relations between the transfer functions of a complex half-band filter $H(z)$ and a real half-band lowpass filter $G(z)$ is given by

$$H(z) = j2G(-jz)$$

- We outline next methods for the design of complex half-band filter based on the above relation

FIR Complex Half-Band Filter

- Let $G(z)$ be the desired FIR real half-band linear-phase lowpass filter of even degree N, with the passband edge at ω_p, stopband edge at ω_s, and passband and stopband ripples of δ, with $\omega_p + \omega_s = \pi$
- The half-band filter $G(z)$ is designed by first designing a wide-band lowpass filter $F(z)$ of degree $N/2$ with a passband from 0 to $2\omega_p$, a transition band from $2\omega_p$ to π, and a passband ripple of 2δ

FIR Complex Half-Band Filter

- The desired half-band filter $G(z)$ is then obtained by forming

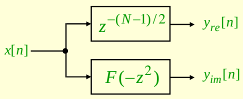
$$G(z) = \tfrac{1}{2}[z^{-N/2} + F(z^2)]$$

- Substituting $H(z) = j2G(-jz)$ in the above equation we obtain

$$H(z) = j[(-jz)^{-N/2} + F(-z^2)]$$
$$= z^{-N/2} + jF(-z^2)$$

FIR Complex Half-Band Filter

- An FIR implementation of the complex half-band filter based on the above decomposition is shown below



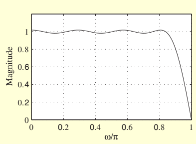
- The linear-phase FIR filter $F(-z^2)$ is thus an approximation to a Hilbert transformer

FIR Complex Half-Band Filter

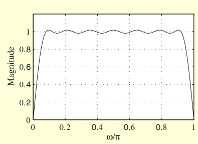
- **Example** – Using the M-file `remez`, we design a wide-band lowpass FIR filter $F(z)$ of degree 13 with a passband from 0 to 0.85π and a small stopband from 0.9π to π
- The magnitude vector used is
  `m = [1 1 0 0]`
- The weight vector used is `wt = [2 0.05]`

FIR Complex Half-Band Filter

- The magnitude responses of the wide-band filter $F(z)$ and the Hilbert transformer $F(-z^2)$ are shown below



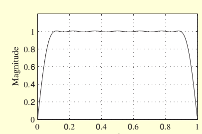
Wide-band lowpass filter



Hilbert transformer

FIR Complex Half-Band Filter

- The FIR Hilbert transformer can also be designed directly using the M-file `remez`
- Figure below shows the magnitude response of a 26-th order FIR Hilbert transformer with a passband from 0.1π to 0.9π designed using `remez`



IIR Complex Half-Band Filter

- We have described earlier the design of a stable IIR real-coefficient half-band filter of odd order can be expressed in the form

$$G(z) = \tfrac{1}{2}[\mathcal{A}_0(z^2) + z^{-1}\mathcal{A}_1(z^2)]$$

where $\mathcal{A}_0(z)$ and $\mathcal{A}_1(z)$ are stable allpass functions

- Substituting the above equation in $H(z) = j2G(-jz)$ we get

$$H(z) = \mathcal{A}_0(-z^2) + j\,z^{-1}\mathcal{A}_1(-z^2)$$

IIR Complex Half-Band Filter

- We have shown earlier that any odd-order elliptic lowpass half-band filter $G(z)$ with a frequency response specification given by

$$1-2\delta_p \le |G(e^{j\omega})| \le 1, \quad \text{for } 0 \le \omega \le \omega_p$$
$$|G(e^{j\omega})| \le \delta_s, \quad \text{for } \omega_s \le \omega \le \pi$$

and satisfying the conditions

$$\omega_p + \omega_s = \pi$$
$$\delta_s^2 = 4\delta_p(1-\delta_p)$$

is a power-symetric transfer function

IIR Complex Half-Band Filter

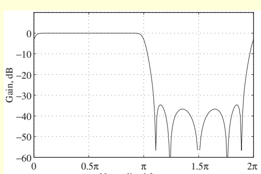
- Moreover, such a transfer function can always be expressed in the form
$$G(z) = \tfrac{1}{2}[\mathcal{A}_0(z^2) + z^{-1}\mathcal{A}_1(z^2)]$$
where $\mathcal{A}_0(z)$ and $\mathcal{A}_1(z)$ are stable allpass functions
- The design of power symmetric stable IIR transfer function can be carried out using Program 13-9

IIR Complex Half-Band Filter

- Consider the design a half-band filter $G(z)$ with a stopband edge at $\omega_s = 0.6\pi$ and a stopband ripple of $\delta_s = 0.016$
- Using Program 13_9, we arrive at the transfer functions of the two allpass sections of $G(z)$ as
$$\mathcal{A}_0(z^2) = \frac{0.236471 + z^{-2}}{1 + 0.236471z^{-2}}$$
$$\mathcal{A}_1(z^2) = \frac{0.7145415 + z^{-2}}{1 + 0.7145415z^{-2}}$$

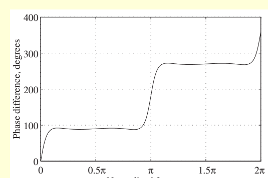
IIR Complex Half-Band Filter

- The gain response of the complex half-band filter $H(z) = \mathcal{A}_0(-z^2) + jz^{-1}\mathcal{A}_1(-z^2)$ is shown below



IIR Complex Half-Band Filter

- Figure below shows the phase difference between the two allpass functions $\mathcal{A}_0(-z^2)$ and $z^{-1}\mathcal{A}_1(-z^2)$ of the complex half-band filter

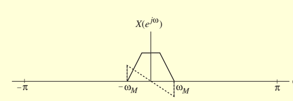


IIR Complex Half-Band Filter

- As expected, the phase difference is 90 degrees for most of the positive frequency range and 270 degrees for most of the negative frequency range

Single-Subband Modulation

- Consider a real low-frequency signal $x[n]$, such as speech or music, with a spectrum $X(e^{j\omega})$ band-limited to ω_M



Single-Subband Modulation

- For efficient transmission over long distances, $x[n]$ is modulated by a very high-frequency carrier signal $\cos\omega_c n$, with the carrier frequency ω_c being less than half of the sampling frequency
- The modulated signal $v[n]$ is then given by

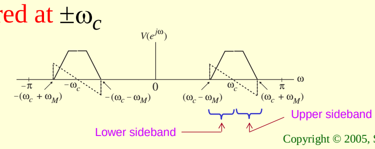
$$v[n] = x[n]\cos\omega_c n$$

Single-Subband Modulation

- The spectrum $V(e^{j\omega})$ of $v[n]$ is given by

$$V(e^{j\omega}) = \tfrac{1}{2}[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})]$$

where $X(e^{j\omega})$ is the spectrum of $x[n]$
- As indicated below, for a band-limited $x[n]$, the spectrum $V(e^{j\omega})$ has a bandwidth of $2\omega_M$ centered at $\pm\omega_c$



Single-Subband Modulation

- By choosing widely separated carrier frequencies, one can modulate a number of low-frequency signals to high-frequency signals, combine them by frequency-division multiplexing, and transmit over a common channel
- The carrier frequencies are chosen appropriately to ensure that there is no overlap in the spectra of the modulated signals when combined by frequency-division multiplexing

Single-Subband Modulation

- At the receiving end, each of the modulated signals is then separated by a bank of bandpass filters with center frequencies corresponding to the different carrier frequencies
- From the figure in Slide 32 it can be seen that, for a low-frequency $x[n]$, the spectrum of the modulated signal $v[n]$ is symmetric with respect to the carrier frequency ω_c

Single-Subband Modulation

- Note: The portion of the spectrum in the frequency range from ω_c to $(\omega_c + \omega_M)$, called the upper sideband, has the same information content as the portion of the spectrum in the frequency range from $(\omega_c + \omega_M)$ to ω_c, called the lower sideband
- Hence, for a more efficient utilization of the channel bandwidth, it is sufficient to transmit either the upper or the lower sideband signal

Single-Subband Modulation

- The single-sideband signal can be generated by modulating the analytic signal whose real and imaginary parts are, respectively, the real signal and its Hilbert transform
- To illustrate this method, let $y[n] = x[n] + \hat{x}[n]$, where $\hat{x}[n]$ is the Hilbert transform of $x[n]$
- Consider

$$s[n] = y[n]e^{j\omega_c n}$$

Single-Subband Modulation

which can be rewritten as

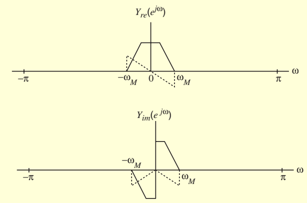
$$s[n] = (y_{re}[n] + j\,y_{re}[n])(\cos\omega_c n + j\sin\omega_c n)$$
$$= (x[n]\cos\omega_c n - \hat{x}[n]\sin\omega_c n)$$
$$+\ j(x[n]\sin\omega_c n + \hat{x}[n]\cos\omega_c n)$$

- From the above, the real and imaginary parts of $s[n]$ are thus given by

$$s_{re}[n] = x[n]\cos\omega_c n - \hat{x}[n]\sin\omega_c n$$
$$s_{im}[n] = x[n]\sin\omega_c n + \hat{x}[n]\cos\omega_c n$$

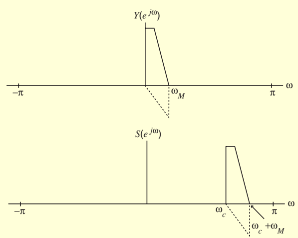
Single-Subband Modulation

- Figures below show the spectra of $x[n]$ and $\hat{x}[n]$



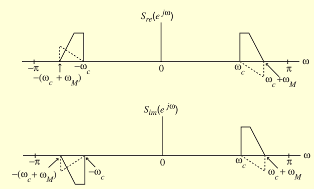
Single-Subband Modulation

- Figures below show the spectra of $y[n]$ and $s[n]$



Single-Subband Modulation

- Figures below show the spectra of $s_{re}[n]$ and $s_{im}[n]$



Single-Subband Modulation

- It follows from the plots given in the previous slide that a single-sideband signal can be generated using either one of the modulation schemes described by the equations in Slide 37
- A block diagram representation of the scheme for generating $s_{re}[n]$ is shown below