Multilevel Filter Banks

- Multiband analysis/synthesis filter banks can also be designed by iterating a 2-channel QMF bank
- Moreover, if the 2-channel QMF bank is of perfect reconstruction type, the generated multiband structure also exhibits perfect reconstruction property

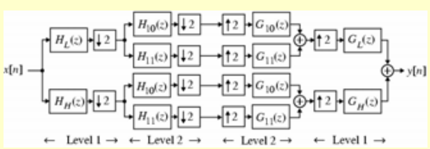
1

Multilevel Filter Banks with Equal Passband Widths

- A 4-channel maximally decimated QMF bank can be designed by inserting a 2-channel maximally decimated QMF bank in each channel of another maximally decimated QMF bank as shown on the next slide

2

Multilevel Filter Banks with Equal Passband Widths



- Since the analysis and synthesis filter banks are formed like a tree, the overall system is often called a tree-structured filter bank

3

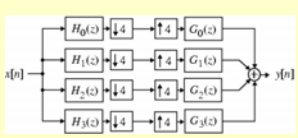
Multilevel Filter Banks with Equal Passband Widths

- In the 4-channel tree-structured filter bank shown on the previous slide, the 2-channel QMF banks in the second level do not have to be identical
- If they are not identical, to compensate for the unequal gains and unequal delays of the 2-channel systems, additional delays of appropriate values need to be inserted to ensure perfect reconstruction

4

Multilevel Filter Banks with Equal Passband Widths

- An equivalent representation of the 4-channel tree-structured filter banks is shown below



5

Multilevel Filter Banks with Equal Passband Widths

- The analysis filters in the equivalent representation are related to those of the parent 2-level QMF bank as follows:

$$H_0(z) = H_L(z)H_{10}(z^2)$$
$$H_1(z) = H_L(z)H_{11}(z^2)$$
$$H_2(z) = H_H(z)H_{10}(z^2)$$
$$H_3(z) = H_H(z)H_{11}(z^2)$$

6

Multilevel Filter Banks with Equal Passband Widths

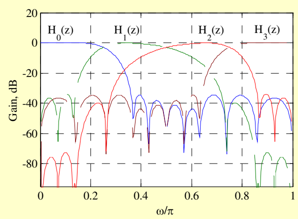
- Likewise, the synthesis filters in the equivalent representation are related to those of the parent 2-level QMF bank as follows:

$$G_0(z) = G_L(z)G_{10}(z^2)$$
$$G_1(z) = G_L(z)G_{11}(z^2)$$
$$G_2(z) = G_H(z)G_{10}(z^2)$$
$$G_3(z) = G_H(z)G_{11}(z^2)$$

Multilevel Filter Banks with Equal Passband Widths

- Example - We design a 4-channel QMF bank by iterating the 2-channel QMF bank based on the filter 12B of Johnston
- Using Program 10_10 we compute the impulse response of the 4 analysis filters and then determine their gain responses as shown on the next slide

Multilevel Filter Banks with Equal Passband Widths



Multilevel Filter Banks with Equal Passband Widths

- Each analysis filter $H_k(z)$ in the equivalent representation is essentially a cascade of two filters, one with a single passband and a single stopband and the other with two passbands and two stopbands
- The passband of the cascade is the frequency range where the passbands of the two filters overlap

Multilevel Filter Banks with Equal Passband Widths

- On the other hand, the stopband of the cascade is formed from three different frequency ranges
- In two of the frequency ranges, the passband of one coincides with the stopband of the other, while in the third range, the two stopbands overlap

Multilevel Filter Banks with Equal Passband Widths

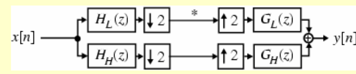
- As a result, the gain responses of the cascade in the three regions are not equal, resulting in an uneven attenuation characteristics
- This type of behavior of the gain response can be seen in the plots shown earlier and should be taken into account in the design of the tree-structured filter bank

Multilevel Filter Banks with Equal Passband Widths

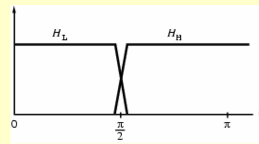
- By continuing the process, QMF banks with more than 4 channels can be easily constructed
- The number of channels resulting from this approach is restricted to be a power of 2, i.e., $L = 2^{\nu}$
- Also, the filters in the analysis (synthesis) branch have passbands of equal width π/L

Filter Banks with Unequal Passband Widths

- Consider the 2-channel maximally decimated QMF bank shown below

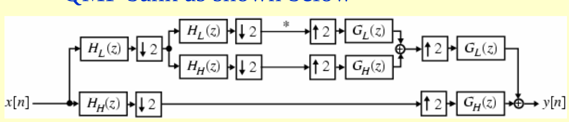


with a typical magnitude response



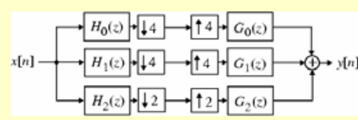
Filter Banks with Unequal Passband Widths

- By inserting another 2-channel maximally decimated QMF bank in the top subband channel at the position marked by a * we arrive at a 3-channel maximally decimated QMF bank as shown below



Filter Banks with Unequal Passband Widths

- The equivalent representation of the generated 3-channel filter bank is shown below



Filter Banks with Unequal Passband Widths

- The analysis and synthesis filters here are given by

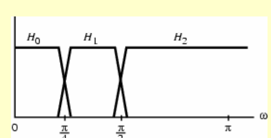
$$H_0(z) = H_L(z)H_L(z^2), \quad G_0(z) = G_L(z)G_L(z^2)$$

$$H_1(z) = H_L(z)H_H(z^2), \quad G_1(z) = G_L(z)G_H(z^2)$$

$$H_2(z) = H_H(z), \quad G_2(z) = G_H(z)$$

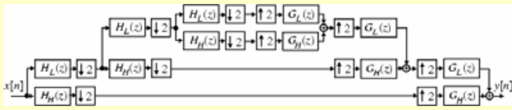
Filter Banks with Unequal Passband Widths

- Typical magnitude responses of the analysis filters of the derived 3-channel QMF bank are shown below



Filter Banks with Unequal Passband Widths

- We can continue this process and generate a 4-channel QMF bank from the 3-channel QMF bank by inserting a 2-channel QMF bank in the top subband channel marked with a *



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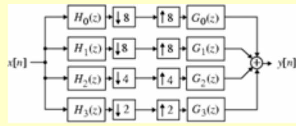
Filter Banks with Unequal Passband Widths

- Its equivalent 4-channel representation is shown below



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Filter Banks with Unequal Passband Widths

- The analysis filters here are given by

$$H_0(z) = H_L(z)H_L(z^2)H_L(z^4)$$
$$H_1(z) = H_L(z)H_L(z^2)H_H(z^4)$$
$$H_2(z) = H_L(z)H_H(z^2)$$
$$H_3(z) = H_H(z)$$

- Corresponding expressions for the synthesis filters can be derived

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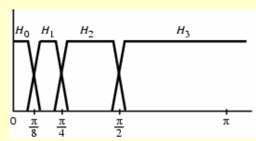
Filter Banks with Unequal Passband Widths

- Figure below shows typical magnitude responses of the 4-channel QMF bank derived from a parent 2-channel QMF bank



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Filter Banks with Unequal Passband Widths

- Because of the unequal passband widths of the analysis and synthesis filters, these QMF structures belong to the class of nonuniform QMF banks
- The tree-structured filter banks have also been referred to as the octave band QMF banks

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Discrete Wavelet Transform

- The function of an octave band analysis filter bank with down-sampling, also called a binary tree, can be considered as a transformation of the input sequence to a set of sub-sequences at the output of the down-samplers
- Consider for example a 4-level binary tree as shown on the next slide

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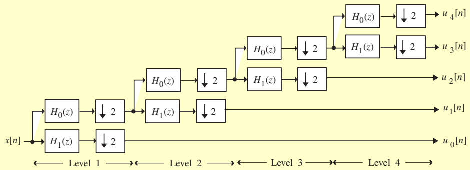
Discrete Wavelet Transform



- The parent analysis filters, $H_0(z)$ and $H_1(z)$, are the analysis filters of a perfect reconstruction 2-channel FIR QMF bank

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Discrete Wavelet Transform

- If the input to this filter bank is a finite-length sequence $x[n]$ of length M, then the output $u_0[n]$ of the highpass analysis filter at the first level is of length $M/2$
- Next, the output $u_0[n]$ of the highpass analysis filter at the second level is of length $M/4$

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Discrete Wavelet Transform

- Continuing this process, we arrive at the outputs of the analysis filters at all levels as shown below along with their lengths
-

Signal	No. of Samples	Approx. Bandwidth
$x[n]$	M	$[0, \pi]$
$u_0[n]$	$M/2$	$[\frac{\pi}{2}, \pi]$
$u_1[n]$	$M/4$	$[\frac{\pi}{4}, \frac{\pi}{2}]$
$u_2[n]$	$M/8$	$[\frac{\pi}{8}, \frac{\pi}{4}]$
$u_3[n]$	$M/16$	$[\frac{\pi}{16}, \frac{\pi}{8}]$
$u_4[n]$	$M/16$	$[0, \frac{\pi}{16}]$

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Discrete Wavelet Transform

- It can be seen from the data given in the previous slide, the total number of output samples is M, independent of the number of levels of the overall octave-band analysis filter bank
- The lowpass analysis filter $H_0(z)$ has a passband of approximate width $[0, \frac{\pi}{2}]$

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Discrete Wavelet Transform

- The highpass analysis filter $H_1(z)$ has a passband of approximate width $[\frac{\pi}{2}, \pi]$
- The lowpass and highpass filters at the second level have passbands of approximate widths given by $[0, \frac{\pi}{4}]$ and $[\frac{\pi}{4}, \frac{\pi}{2}]$, respectively
- In a similar manner, the approximate widths of the passbands of the analysis filters at each succeeding levels can be determined

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Discrete Wavelet Transform

- The table in Slide 27 shows the approximate bandwidths of all signals at the outputs of the down-samplers of the 4-level binary tree shown in Slide 25
- An equivalent representation with a single down-sampler in each channel is indicated in the next slide

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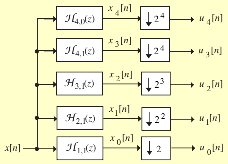
Discrete Wavelet Transform



- The expressions for the transfer functions shown above in terms of the analysis filters are given in the next slide

Discrete Wavelet Transform

$\mathcal{H}_{1,1}(z) = H_1(z)$
$\mathcal{H}_{2,1}(z) = H_0(z)H_1(z^2)$
$\mathcal{H}_{3,1}(z) = H_0(z)H_0(z^2)H_1(z^4)$
$\mathcal{H}_{4,1}(z) = H_0(z)H_0(z^2)H_0(z^4)H_1(z^8)$
$\mathcal{H}_{4,0}(z) = H_0(z)H_0(z^2)H_0(z^4)H_0(z^8)$

- In the general case, the transfer functions to the two outputs at level k are given by

$$\mathcal{H}_{k,1}(z) = \left[\prod_{i=0}^{k-2} H_0(z^{2^i})\right] H_1(z^{2^{k-1}})$$

$$\mathcal{H}_{k,0}(z) = \prod_{i=0}^{k-1} H_0(z^{2^i})$$

Discrete Wavelet Transform

- The process of generating the set of output sub-sequences $u_k[n], 0 \le k \le L$, from the finite-length input sequence $x[n]$ using the analysis filters of an L-level octave-band perfect reconstruction QMF bank is known as the discrete wavelet transform (DWT)

Discrete Wavelet Transform

- More precisely, the wavelet at level k is related to the impulse response $\mathcal{h}_{k,1}[n]$ of the filter $\mathcal{H}_{k,1}(z)$ for the k-th bandpass output
- The basic impulse response $\mathcal{h}_{1,1}[n]$ given by the inverse z-transform of $\mathcal{H}_{k,1}(z) = H_1(z)$ is called the mother wavelet

Discrete Wavelet Transform

- The impulse response $\mathcal{h}_{k,0}[n]$ of the lowpass filter $\mathcal{H}_{k,0}(z)$ is called the scaling function at level k
- The input-output of the k-th channel before down-sampling in the z-domain of an L-level octave band perfect reconstruction QMF bank is related through

$$X_k(z) = \mathcal{H}_{k,1}(z)X(z), \quad 0 \le k \le L-1$$
$$X_k(z) = \mathcal{H}_{k,0}(z)X(z), \quad k = L$$

Discrete Wavelet Transform

- Or equivalently, in the time-domain, through

$$x_k[n] = \sum_{m=-\infty}^{\infty} \mathcal{h}_{k,1}[n-m]x[m], \quad 0 \le k \le L-1$$

$$x_k[n] = \sum_{m=-\infty}^{\infty} \mathcal{h}_{k,0}[n-m]x[m], \quad k = L$$

Discrete Wavelet Transform

- The equivalent representation of the k-th bandpass channel of the binary analysis tree, for $0 \le k \le L-1$, is a factor-of-2^{k+1} decimator with a decimation filter $\mathcal{H}_{k,1}(z)$
- The equivalent representation of the L-th bandpass channel of the binary analysis tree is a factor-of-2^L decimator with a decimation filter $\mathcal{H}_{L,0}(z)$

Discrete Wavelet Transform

- Thus, the DWT $u_k[n], 0 \le k \le L$, of $x[n]$ is given by

$$u_k[n] = \sum_{m=-\infty}^{\infty} \mathcal{h}_{k,1}[2^{k+1}n - m]x[m], \quad 0 \le k \le L-1$$

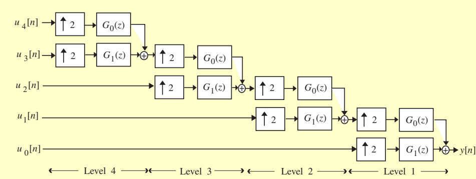
$$u_k[n] = \sum_{m=-\infty}^{\infty} \mathcal{h}_{k,0}[2^{k}n - m]x[m], \quad k = L$$

Discrete Wavelet Transform

- The reverse process of reconstructing the output sequence $y[n]$, which is a replica of the input $x[n]$, from the sub-sequences using the synthesis filters of the L-level octave band perfect reconstruction QMF bank is the inverse discrete wavelet transform (IDWT)

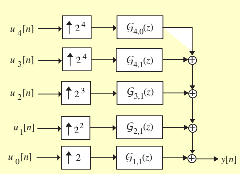
Discrete Wavelet Transform

- The 4-level synthesis binary tree corresponding to the analysis binary tree is shown below:



Discrete Wavelet Transform

- An equivalent representation of the 4-level octave-band synthesis filter bank using a single up-sampler in each path is shown below:



Discrete Wavelet Transform

- The transfer functions from each input (after up-sampling) to the output of the L-level octave-band synthesis filter bank are given by

$$\mathcal{G}_{1,1}(z) = G_1(z)$$
$$\mathcal{G}_{2,1}(z) = G_0(z)G_1(z^2)$$
$$\mathcal{G}_{3,1}(z) = G_0(z)G_0(z^2)G_1(z^4)$$
$$\mathcal{G}_{4,1}(z) = G_0(z)G_0(z^2)G_0(z^4)G_1(z^8)$$
$$\mathcal{G}_{4,0}(z) = G_0(z)G_0(z^2)G_0(z^4)G_0(z^8)$$

Discrete Wavelet Transform

- In the general case, the transfer functions to the output from the two inputs at level k of the binary tree after up-sampling are given by

$$\mathcal{G}_{k,1}(z)=\left[\prod_{i=0}^{k-2} G_0(z^{2^i})\right]G_1(z^{2^{k-1}})$$
$$\mathcal{G}_{k,0}(z)=\prod_{i=0}^{k-1} G_0(z^{2^i})$$

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Discrete Wavelet Transform

- Under the perfect reconstruction condition, the output of the L-channel synthesis filter bank is identical to the input of the analysis filter bank, that is, $y[n] = x[n]$
- In the z-domain, we can express the output in terms of the inputs to the synthesis filter bank (after up-sampling) through

$$Y(z)=\mathcal{G}_{1,1}(z)U_0(z^2)+\mathcal{G}_{2,1}(z)U_1(z^{2^2})+\cdots$$
$$+\mathcal{G}_{L,1}(z)U_{L-1}(z^{2^L})+\mathcal{G}_{L,0}(z)U_L(z^{2^L})$$

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Discrete Wavelet Transform

- Here, the equivalent representation of the k-th bandpass channel, $0 \le k \le L-1$, of the synthesis binary tree is a factor-of-2^{k+1} interpolator with an interpolation filter $\mathcal{G}_{k,1}(z)$ and that of the L-th lowpass channel is a factor-of-2^L interpolator with an interpolation filter $\mathcal{G}_{L,0}(z)$

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Discrete Wavelet Transform

- In the time-domain, the output $y[n]$ as a function of the inputs $u_k[n]$ of the synthesis binary tree is given by

$$y[n]=\sum_{k=0}^{L-1}\sum_{m=-\infty}^{\infty} g_{k,1}[n-2^{k+1}m]u_k[m]$$
$$+\sum_{m=-\infty}^{\infty} g_{L,0}[n-2^L m]u_L[m]$$

where $g_{k,\ell}[n]$, $\ell=0,1$, is the impulse response of $\mathcal{G}_{k,\ell}(z)$

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Discrete Wavelet Transform

- Using the notation

$$\eta_{k,m}[n]=g_{k,1}[n-2^{k+1}m], \quad 0\le k\le L-1$$
$$\eta_{L,m}[n]=g_{L,0}[n-2^L m]$$

the expression for the IDWT can be compactly rewritten as

$$y[n]=\sum_{k=0}^{L}\sum_{m=-\infty}^{\infty}\eta_{k,m}[n]u_k[m]$$

- In the above expression, $u_k[m]$ are the wavelet coefficients of $x[n]$ with respect to the basis functions $\eta_{k,m}[n]$

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Biorthogonal Wavelets

- The biorthogonal wavelets are generated from an octave-band filter bank designed from a 2-channel biorthogonal perfect reconstruction QMF bank
- The simplest type of the biorthogonal wavelets are the Haar wavelets generated using the Haar filters

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Biorthogonal Wavelets

- In the case of a 4-level octave-band analysis filter bank, the transfer functions of the analysis filters in the tree are given by

$$\mathcal{H}_{1,1}(z) = \tfrac{1}{\sqrt{2}}(1 - z^{-1})$$

$$\mathcal{H}_{2,1}(z) = \tfrac{1}{2}(1 + z^{-1})(1 - z^{-2})$$

$$\mathcal{H}_{3,1}(z) = \tfrac{1}{2\sqrt{2}}(1 + z^{-1})(1 + z^{-2})(1 - z^{-4})$$

$$\mathcal{H}_{4,1}(z) = \tfrac{1}{4}(1 + z^{-1})(1 + z^{-2})(1 + z^{-4})(1 - z^{-8})$$

$$\mathcal{H}_{4,0}(z) = \tfrac{1}{4}(1 + z^{-1})(1 + z^{-2})(1 + z^{-4})(1 + z^{-8})$$

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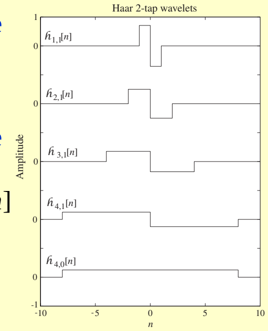
Biorthogonal Wavelets

- A plot of the impulse responses (shown as continuous plots) of the analysis filter transfer functions are shown on the right
- Note: Shape of $\hat{h}_{1,1}[n]$ is independent of scale and is the mother wavelet



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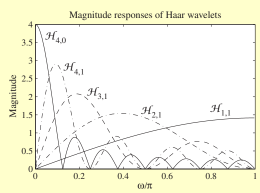
Biorthogonal Wavelets

- The magnitude responses of the 4-level tree of Haar filters are shown below



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Biorthogonal Wavelets

- Haar wavelets exhibit abrupt transitions at the middle and at both ends
- These transitions often lead to visible blocking artifacts in the images reconstructed from their compressed versions
- As a result, wavelets with gradual transitions are usually preferred for image compression applications

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Biorthogonal Wavelets

- LeGall 3/5-tap wavelets are generated from the LeGall 3/5-tap analysis filter pair:

$$H_0(z) = \tfrac{1}{2}(1 + 2z^{-1} + z^{-2})$$

$$H_1(z) = \tfrac{1}{8}(-1 + 2z^{-1} + 6z^{-2} + 2z^{-3} - z^{-4})$$

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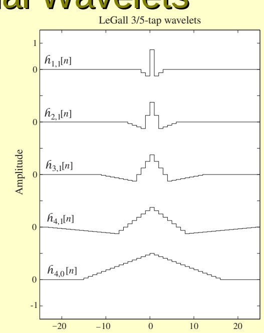
Biorthogonal Wavelets

- The plot of the impulse responses (shown as continuous plots) of the LeGall 3/5-tap wavelets is shown on the right



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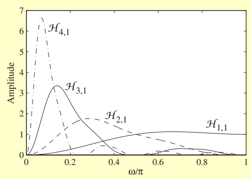
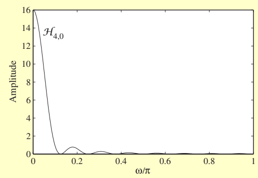
Biorthogonal Wavelets

- The magnitude responses of the 4-level tree of LeGall 3/5-tap filters are shown below





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Biorthogonal Wavelets

- The scaling function $\hat{h}_{4,0}[n]$ of the LeGall 3/5-tap wavelets, shown in Slide 54, converges to a pure triangular pulse, after many levels in which case the wavelets are basically determined by the superposition of two triangular pulses

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Orthogonal Wavelets

- The orthogonal wavelets are generated from an octave-band filter bank designed from a 2-channel orthogonal perfect reconstruction QMF bank
- For example, the Daubechies 4/4-tap wavelets are generated from Daubechies 4/4-tap analysis filter pairs

$$H_0(z) = \tfrac{1}{8}(1 + 3z^{-1} + 3z^{-2} + z^{-3})$$

$$H_1(z) = \tfrac{1}{2}(-1 - 3z^{-1} + 3z^{-2} + z^{-3})$$

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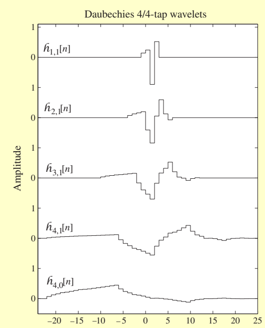
Orthogonal Wavelets

- The impulse responses (shown as continuous plots) of the 4-level tree of the Daubechies 4/4-tap filters are shown on the right



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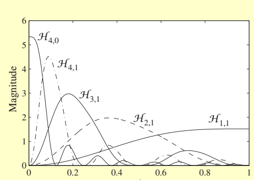
Orthogonal Wavelets

- The magnitude responses of the 4-level tree of Daubechies 4/4-tap filters are shown below



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Orthogonal Wavelets

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