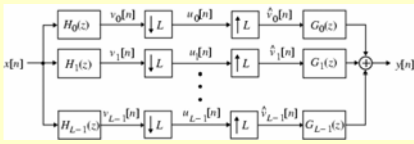
L-Channel QMF Banks

- The basic structure of the L-channel QMF bank is shown below



- The expressions for the z-transforms of various intermediate signals in the above structure are given by

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L-Channel QMF Banks

$$V_k(z) = H_k(z)X(z)$$

$$U_k(z) = \frac{1}{L}\sum_{\ell=0}^{L-1} H_k(z^{1/L}W_L^{\ell})X(z^{1/L}W_L^{\ell})$$

$$\hat{V}_k(z) = U_k(z^L)$$

where $0 \le k \le L-1$

- Define the vector of down-sampled subband signals as

$$\mathbf{u}(z) = [U_0(z) \;\; U_1(z) \;\; \cdots \;\; U_{L-1}(z)]^T$$

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L-Channel QMF Banks

- Define the modulation vector of the input signals as

$$\mathbf{x}^{(m)}(z) = [X(z) \;\; X(zW_L) \;\; \cdots \;\; X(zW_L^{L-1})]^T$$

- Define the analysis filter bank modulation matrix as

$$\mathbf{H}^{(m)}(z) = \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{L-1}(z) \\ H_0(zW_L) & H_1(zW_L) & \cdots & H_{L-1}(zW_L) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(zW_L^{L-1}) & H_1(zW_L^{L-1}) & \cdots & H_{L-1}(zW_L^{L-1}) \end{bmatrix}$$

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L-Channel QMF Banks

- Then we can write the set of L equations

$$U_k(z) = \tfrac{1}{L}\textstyle\sum_{\ell=0}^{L-1} H_k(z^{1/L}W_L^{\ell})X(z^{1/L}W_L^{\ell}),$$
$$0 \le k \le L-1$$

as

$$\mathbf{u}(z) = \tfrac{1}{L}[\mathbf{H}^{(m)}(z^{1/L})]^T \mathbf{x}^{(m)}(z^{1/L})$$

- The output of the QMF bank is given by

$$Y(z) = \sum_{k=0}^{L-1} G_k(z)\hat{V}_k(z)$$

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L-Channel QMF Banks

- In matrix form we can write

$$Y(z) = \mathbf{g}^T(z)\mathbf{u}(z^L)$$

where

$$\mathbf{g}(z) = [G_0(z) \;\; G_1(z) \;\; \cdots \;\; G_{L-1}(z)]^T$$

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Alias-Free *L*-Channel QMF Banks

- From the output equation

$$Y(z) = \mathbf{g}^T(z)\mathbf{u}(z^L)$$

the modulated versions of the output signal are given by

$$Y(zW_L^k) = \mathbf{g}^T(zW_L^k)\mathbf{u}(z^L W_L^{kL}) = \mathbf{g}^T(zW_L^k)\mathbf{u}(z^L),$$
$$0 \le k \le L-1$$

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Alias-Free L-Channel QMF Banks

- Define the modulation vector of the output signal as

$$\mathbf{y}^{(m)}(z) = [Y(z)\ \ Y(zW_L)\ \ \cdots\ \ Y(zW_L^{L-1})]^T$$

- Define the synthesis filter bank modulation matrix as

$$\mathbf{G}^{(m)}(z) = \begin{bmatrix} G_0(z) & G_1(z) & \cdots & G_{L-1}(z) \\ G_0(zW_L) & G_1(zW_L) & \cdots & G_{L-1}(zW_L) \\ \vdots & \vdots & \ddots & \vdots \\ G_0(zW_L^{L-1}) & G_1(zW_L^{L-1}) & \cdots & G_{L-1}(zW_L^{L-1}) \end{bmatrix}$$

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Alias-Free L-Channel QMF Banks

- Then the modulation vector of the output signal can be expressed as

$$\mathbf{y}^{(m)}(z) = \mathbf{G}^{(m)}(z)\mathbf{u}(z^L)$$

- Combining the above and

$$\mathbf{u}(z) = \tfrac{1}{L}[\mathbf{H}^{(m)}(z^{1/L})]^T \mathbf{x}^{(m)}(z^{1/L})$$

we arrive at

$$\mathbf{y}^{(m)}(z) = \tfrac{1}{L}\mathbf{G}^{(m)}(z)[\mathbf{H}^{(m)}(z)]^T \mathbf{x}^{(m)}(z)$$

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Alias-Free L-Channel QMF Banks

- Using the notation

$$\mathbf{T}(z) = \tfrac{1}{L}\mathbf{G}^{(m)}(z)[\mathbf{H}^{(m)}(z)]^T$$

we can write

$$\mathbf{y}^{(m)}(z) = \mathbf{T}(z)\mathbf{x}^{(m)}(z)$$

- $\mathbf{T}(z)$ is called the transfer matrix relating the input signal $X(z)$ and its modulated versions $X(zW_L^k)$ with the output signal $Y(z)$ and its modulated versions $Y(zW_L^k)$

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Alias-Free L-Channel QMF Banks

- The filter bank is alias-free if the transfer matrix $\mathbf{T}(z)$ is a diagonal matrix of the form

$$\mathbf{T}(z) = \mathrm{diag}[T(z)\ \ T(zW_L)\ \ \cdots\ \ T(zW_L^{L-1})]$$

- The first element $T(z)$ of the above diagonal matrix is called the distortion transfer function of the L-channel filter bank

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Alias-Free L-Channel QMF Banks

- Substituting

$$V_k(z) = H_k(z)X(z)$$

$$U_k(z) = \frac{1}{L}\sum_{\ell=0}^{L-1} H_k(z^{1/L}W_L^\ell)X(z^{1/L}W_L^\ell)$$

$$\hat{V}_k(z) = U_k(z^L)$$

in

$$Y(z) = \sum_{k=0}^{L-1} G_k(z)\hat{V}_k(z)$$

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Alias-Free L-Channel QMF Banks

we arrive at

$$Y(z) = \textstyle\sum_{\ell=0}^{L-1} a_\ell(z)X(zW_L^\ell)$$

where

$$a_\ell(z) = \tfrac{1}{L}\textstyle\sum_{k=0}^{L-1} H_k(zW_L^\ell)G_k(z), \quad 0 \le \ell \le L-1$$

- On the unit circle the term $X(zW_L^\ell)$ becomes

$$X(e^{j\omega}W_L^\ell) = X(e^{j(\omega - 2\pi\ell/L)})$$

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Alias-Free L-Channel QMF Banks

- Thus, from

$$Y(z) = \sum_{\ell=0}^{L-1} a_\ell(z) X(zW_L^\ell)$$

we observe that the output spectrum $Y(e^{j\omega})$ is a weighted sum of $X(e^{j\omega})$ and its uniformly shifted versions $X(e^{j(\omega - 2\pi\ell/L)})$ for $\ell = 1, 2, \ldots, L-1$ which are caused by the sampling rate alteration operations

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Alias-Free L-Channel QMF Banks

- The term $X(zW_L^\ell)$ is called the ℓ-th aliasing term, with $a_\ell(z)$ representing its gain at the output
- In general, the L-channel QMF bank is a linear, time-varying system with a period L
- It follows from $Y(z) = \sum_{\ell=0}^{L-1} a_\ell(z) X(zW_L^\ell)$ that the aliasing effect at the output can be completely eliminated if and only if

$$a_\ell(z) = 0, \quad 1 \le \ell \le L-1$$

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Alias-Free L-Channel QMF Banks

- Note: The aliasing cancellation condition given above must hold for all possible inputs
- If the aliasing cancellation condition holds then the L-channel QMF bank becomes a linear, time-invariant system with an input-output relation given by

$$Y(z) = T(z)X(z)$$

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Alias-Free L-Channel QMF Banks

- The distortion transfer function $T(z)$ is given by

$$T(z) = a_0(z) = \tfrac{1}{L} \sum_{k=0}^{L-1} H_k(z) G_k(z)$$

- If $T(z)$ has a constant magnitude, then the L-channel QMF bank is magnitude-preserving
- If $T(z)$ has a linear phase, then the L-channel QMF bank is phase-preserving
- If $T(z)$ is a pure delay, then it is a perfect reconstruction filter bank

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Alias-Free L-Channel QMF Banks

- Define

$$\mathbf{A}(z) = [a_0(z) \quad a_1(z) \quad \cdots \quad a_{L-1}(z)]$$

- Then

$$a_\ell(z) = \tfrac{1}{L} \sum_{k=0}^{L-1} H_k(zW_L^\ell) G_k(z), \quad 0 \le \ell \le L-1$$

can be expressed as

$$L \cdot \mathbf{A}(z) = \mathbf{H}^{(m)}(z) \mathbf{g}(z)$$

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Alias-Free L-Channel QMF Banks

- The aliasing cancellation condition can now be rewritten as

$$\mathbf{H}^{(m)}(z) \mathbf{g}(z) = \mathbf{t}(z)$$

where

$$\mathbf{t}(z) = [L a_0(z) \quad 0 \quad \cdots \quad 0]^T$$

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Alias-Free L-Channel QMF Banks

- Hence, knowing the set of analysis filters $\{H_k(z)\}$, we can determine the desired set of synthesis filters $\{G_k(z)\}$ as
$$\mathbf{g}(z) = [\mathbf{H}^{(m)}(z)]^{-1}\mathbf{t}(z)$$
provided $[\det \mathbf{H}^{(m)}(z)] \neq 0$
- Moreover, a perfect reconstruction QMF bank results if we set $T(z) = z^{-n_o}$ in the expression for $\mathbf{t}(z)$

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Alias-Free L-Channel QMF Banks

- In practice, the above approach is difficult to carry out for a number of reasons
- A more practical solution to the design of a perfect reconstruction QMF bank is based on a polyphase representation

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Polyphase Representation

- Consider the L-band Type I polyphase representation of the k-th analysis filter:
$$H_k(z) = \sum_{\ell=0}^{L-1} z^{-\ell} E_{k\ell}(z^L), \quad 0 \le k \le L-1$$
- A matrix representation of the above set of equations is given by
$$\mathbf{h}(z) = \mathbf{E}(z^L)\mathbf{e}(z)$$
where
$$\mathbf{h}(z) = [H_0(z) \;\; H_1(z) \;\; \cdots \;\; H_{L-1}(z)]^T$$

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Polyphase Representation

$$\mathbf{e}(z) = [1 \;\; z^{-1} \;\; \cdots \;\; z^{-(L-1)}]^T$$
and
$$\mathbf{E}(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) & \cdots & E_{0,L-1}(z) \\ E_{10}(z) & E_{11}(z) & \cdots & E_{1,L-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ E_{L-1,0}(z) & E_{L-1,1}(z) & \cdots & E_{L-1,L-1}(z) \end{bmatrix}$$

- $\mathbf{E}(z)$ is called the Type I polyphase component matrix

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Polyphase Representation

- Likewise, we can represent the L synthesis filters in a L-band Type II polyphase form:
$$G_k(z) = \sum_{\ell=0}^{L-1} z^{-(L-1-\ell)} R_{\ell k}(z^L), \quad 0 \le k \le L-1$$
- In matrix form the above set of equations can be rewritten as
$$\mathbf{g}^T(z) = z^{-(L-1)}\tilde{\mathbf{e}}(z)\mathbf{R}(z^L)$$
where
$$\mathbf{g}(z) = [G_0(z) \;\; G_1(z) \;\; \cdots \;\; G_{L-1}(z)]^T$$

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Polyphase Representation

$$\mathbf{e}(z) = [1 \;\; z \;\; \cdots \;\; z^{L-1}] = \mathbf{e}^T(z^{-1})$$
and
$$\mathbf{R}(z) = \begin{bmatrix} R_{00}(z) & R_{01}(z) & \cdots & R_{0,L-1}(z) \\ R_{10}(z) & R_{11}(z) & \cdots & R_{1,L-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ R_{L-1,0}(z) & R_{L-1,1}(z) & \cdots & R_{L-1,L-1}(z) \end{bmatrix}$$

- $\mathbf{R}(z)$ is called the Type II polyphase component matrix

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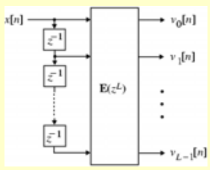
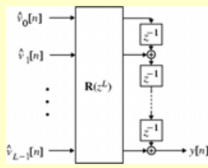
Polyphase Representation

- The polyphase representations of the L-channel analysis and the L-channel synthesis filter banks are shown below





Analysis filter bank Synthesis filter bank

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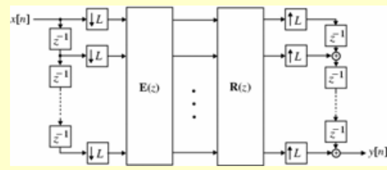
Polyphase Representation

- Substituting the polyphase representations of the analysis and synthesis filter banks in the original structure of the L-channel QMF bank, and making use of the cascade equivalences we arrive at



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Polyphase Representation

- From

$$\mathbf{H}^{(m)}(z) = \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{L-1}(z) \\ H_0(zW_L) & H_1(zW_L) & \cdots & H_{L-1}(zW_L) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(zW_L^{L-1}) & H_1(zW_L^{L-1}) & \cdots & H_{L-1}(zW_L^{L-1}) \end{bmatrix}$$

and $\mathbf{h}(z) = [H_0(z) \;\; H_1(z) \;\; \cdots \;\; H_{L-1}(z)]^T$ it can be seen that

$$[\mathbf{H}^{(m)}(z)]^T = [\mathbf{h}(z) \;\; \mathbf{h}(zW_L) \;\; \cdots \;\; \mathbf{h}(zW_L^{L-1})]$$

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Polyphase Representation

- Making use of $\mathbf{h}(z) = \mathbf{E}(z^L)\mathbf{e}(z)$ in the previous equation we get

$$[\mathbf{H}^{(m)}(z)]^T = \mathbf{E}(z^L)[\mathbf{e}(z) \;\; \mathbf{e}(zW_L) \;\; \cdots \;\; \mathbf{e}(zW_L^{L-1})]$$

- Now, from $\mathbf{e}(z) = [1 \;\; z^{-1} \;\; \cdots \;\; z^{-(L-1)}]^T$ we have

$$\mathbf{e}(zW_L^k) = \Delta(z) \begin{bmatrix} 1 \\ W_L^{-k} \\ \vdots \\ W_L^{-k(L-1)} \end{bmatrix}$$

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Polyphase Representation

where we have used the notation

$$\Delta(z) = \mathrm{diag}[1 \;\; z^{-1} \;\; \cdots \;\; z^{-(L-1)}]$$

- Making use of the above notation in

$$[\mathbf{H}^{(m)}(z)]^T = \mathbf{E}(z^L)[\mathbf{e}(z) \;\; \mathbf{e}(zW_L) \;\; \cdots \;\; \mathbf{e}(zW_L^{L-1})]$$

we arrive at

$$\mathbf{H}(z) = \mathbf{D}^{\dagger}\Delta(z)\mathbf{E}^T(z^L)$$

where $\mathbf{D}^{\dagger}$ is the conjugate transpose of the $L \times L$ DFT matrix $\mathbf{D}$

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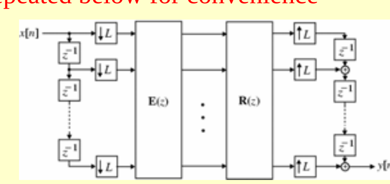
Condition for Perfect Reconstruction

- Consider the L-channel QMF structure repeated below for convenience



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Condition for Perfect Reconstruction

- Assume that the polyphase component matrices satisfy the relation

$$\mathbf{R}(z)\mathbf{E}(z) = c\mathbf{I}$$

where $\mathbf{I}$ is an $L \times L$ identity matrix and c is a constant

- Then the QMF structure on the previous slide reduces to the one shown on the next slide

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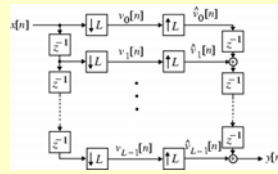
Condition for Perfect Reconstruction



- Note: The structure can be considered as a special case of the most general L-channel QMF bank shown earlier if we set
$H_k(z) = z^{-k}, \;\; G_k(z) = z^{-(L-1-k)}, \;\; 0 \le k \le L-1$

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Condition for Perfect Reconstruction

- Substituting

$$H_k(z) = z^{-k}, \;\; G_k(z) = z^{-(L-1-k)}, \;\; 0 \le k \le L-1$$

in

$$a_\ell(z) = \tfrac{1}{L}\textstyle\sum_{k=0}^{L-1} H_k(zW_L^\ell)G_k(z), \;\; 0 \le \ell \le L-1$$

we get

$$a_\ell(z) = z^{-(L-1)}\left(\tfrac{1}{L}\textstyle\sum_{\ell=0}^{L-1} W_L^{-\ell k}\right), \;\;\; 0 \le \ell \le L-1$$

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Condition for Perfect Reconstruction

- Now, $\tfrac{1}{L}\sum_{\ell=0}^{L-1} W_L^{-\ell k} = \begin{cases} 1, & \ell = 0 \\ 0, & 1 \le \ell \le L-1 \end{cases}$

- Hence, from the last equation on the previous slide it follows that

$$a_0(z) = 1, \;\; a_\ell(z) = 0 \;\; \text{for } \ell \ne 0$$

- As a result, $T(z) = z^{-(L-1)}$ or in other words, the simplified QMF structure satisfies the perfect reconstruction property

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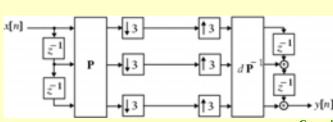
Condition for Perfect Reconstruction

- The analysis and synthesis filters of the perfect reconstruction L-channel QMF bank can be easily determined from known polyphase component matrices
- Example - The structure shown below is by construction a perfect reconstruction filter bank



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Condition for Perfect Reconstruction

- The output of the filter bank is simply

$$y[n] = d\,x[n-2]$$

- Note: In this structure $\mathbf{E}(z^3) = \mathbf{P}$ and $\mathbf{R}(z^3) = d\mathbf{P}^{-1}$
- Consider

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

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Condition for Perfect Reconstruction

- From $\mathbf{h}(z) = \mathbf{E}(z^3)\mathbf{e}(z)$ we get

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} = \begin{bmatrix} E_{00}(z) & E_{01}(z) & E_{02}(z) \\ E_{10}(z) & E_{11}(z) & E_{12}(z) \\ E_{20}(z) & E_{21}(z) & E_{22}(z) \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}$$

- Hence,

$$H_0(z) = 1 + z^{-1} + z^{-2}, \quad H_1(z) = 1 - z^{-1} + z^{-2},$$
$$H_2(z) = 1 - z^{-2}$$

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Condition for Perfect Reconstruction

- With $d = 4$ we have $d\mathbf{P}^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -2 & 0 \\ 1 & 1 & -2 \end{bmatrix}$
- Then from $\mathbf{g}^T(z) = z^{-(L-1)}\tilde{\mathbf{e}}(z)\mathbf{R}(z^3)$ we get

$$\begin{bmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \end{bmatrix} = \begin{bmatrix} z^{-2} & z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & -2 & 0 \\ 1 & 1 & -2 \end{bmatrix}$$

which leads to

$$G_0(z) = 1 + 2z^{-1} + z^{-2}, \quad G_1(z) = 1 - 2z^{-1} + z^{-2},$$
$$G_2(z) = -2 + 2z^{-2}$$

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Polyphase Representation

- For a given L-channel analysis filter bank, the polyphase component matrix $\mathbf{E}(z)$ is known $\mathbf{E}(z)\mathbf{R}(z) = c\mathbf{I}$,
- Hence, a perfect reconstruction L-channel QMF bank can be designed by constructing a synthesis filter bank with a polyphase component matrix

$$\mathbf{R}(z) = [\mathbf{E}(z)]^{-1}$$

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Polyphase Representation

- In general, it is not easy to compute the inverse of a rational $L \times L$ matrix
- An alternative elegant approach is to design the analysis filter bank with an invertible polyphase matrix $\mathbf{E}(z)$
- For example, $\mathbf{E}(z)$ can be chosen to be a paraunitary matrix satisfying the condition

$$\tilde{\mathbf{E}}(z)\mathbf{E}(z) = c\mathbf{I}, \qquad \text{for all } z$$

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Polyphase Representation

- Note: $\tilde{\mathbf{E}}(z)$ is the paraconjugate of $\mathbf{E}(z)$ given by the transpose of $\mathbf{E}(z^{-1})$, with each coefficient replaced by its conjugate
- A perfect reconstruction L-channel QMF bank is then obtained by choosing

$$\mathbf{R}(z) = \tilde{\mathbf{E}}(z)$$

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Polyphase Representation

- For the design of a perfect reconstruction L-channel QMF bank, the matrix $\mathbf{E}(z)$ can be expressed in a product form

$$\mathbf{E}(z) = \mathbf{E}_R(z)\mathbf{E}_{R-1}(z)\cdots\mathbf{E}_1(z)\mathbf{E}_0$$

where $\mathbf{E}_0$ is a constant unitary matrix, and

$$\mathbf{E}_\ell(z) = \mathbf{I} - \mathbf{v}_\ell[\mathbf{v}_\ell^*]^T + z^{-1}\mathbf{v}_\ell[\mathbf{v}_\ell^*]^T$$

- In the above $\mathbf{v}_\ell$ is a column vector of order L with unit norm, i.e., $[\mathbf{v}_\ell^*]^T\mathbf{v}_\ell = 1$

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Polyphase Representation

- With $\mathbf{E}(z)$ expressed in the product form, one can set up an appropriate objective function that can be minimized to arrive at a set of analysis filters meeting the desired specifications
- To this end, a suitable objective function is given by

$$\Phi = \sum_{k=0}^{L-1} \int_{k-\text{th stopband}} \left|H_k(e^{j\omega})\right|^2 d\omega$$

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Polyphase Representation

- The optimization parameters are the elements of $\mathbf{v}_\ell$ and $\mathbf{E}_0$
- Example - Consider the design of a 3-channel FIR perfect reconstruction QMF bank with a passband width $\pi/3$
- The passband width of the lowpass filter is from 0 to $\pi/3$, that of the bandpass filter is from $\pi/3$ to $2\pi/3$, and that of the highpass filter is from $2\pi/3$ to π

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Polyphase Representation

- The objective function to be minimized here is thus of the form

$$\Phi = \int_{\frac{\pi}{3}+\varepsilon}^{\pi} \left|H_0(e^{j\omega})\right|^2 d\omega + \int_{0}^{\frac{\pi}{3}-\varepsilon} \left|H_1(e^{j\omega})\right|^2 d\omega + \int_{\frac{2\pi}{3}+\varepsilon}^{\pi} \left|H_1(e^{j\omega})\right|^2 d\omega + \int_{0}^{\frac{2\pi}{3}-\varepsilon} \left|H_2(e^{j\omega})\right|^2 d\omega$$

- The gain responses of the 3 analysis filters of length 15 are shown on the next slide

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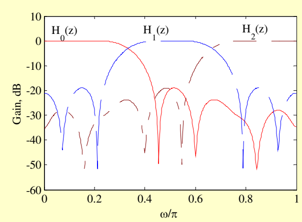
Polyphase Representation



- The coefficients of the corresponding synthesis filters are given by

$$g_k[n] = h_k[14-n], \ \ k = 1,2,3$$

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Multiplexing Schemes

- In most countries including the United States, two types of multiplexing schemes are used to transmit multiple low-frequency voice signals over a wide-band channel

  (1) Frequency-division multiplex (FDM) system

  (2) Time-division multiplex (TDM) system

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Frequency-Division Multiplex System

- In this system, multiple analog voice signals are first modulated by single-sideband (SSB) modulators onto several subcarriers, combined, and transmitted simultaneously over a common wide-band channel
- To avoid cross-talk, the subcarriers are chosen to ensure that the spectra of the modulated signals do not overlap

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Frequency-Division Multiplex System

- At the receiving end, the modulated subcarrier signals are separated by analog bandpass filters and demodulated to reconstruct the individual voice signals

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Time-Division Multiplex System

- In this system, the voice signals are first converted into digital signals by sampling and A/D conversion
- The samples of the digital signals are then time-interleaved by a digital multiplexer, and the combined signal is transmitted

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Time-Division Multiplex System

- At the receiving end, the digital voice signals are separated by a digital multiplexer
- The individual digital signal is then passed through a D/A converter and an analog reconstruction filter to recover the original analog voice signal

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FDM and TDM Systems

- The TDM system is usually employed for short-haul communication
- The FDM system is preferred for long-haul transmission
- Until the telephone service becomes all digital, it is necessary to translate signals between these two formats which is achieved by the transmultiplexer

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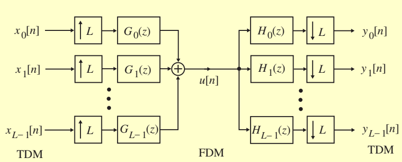
Transmultiplexer

- The transmultiplexer is a multi-input, multi-output, multirate structure as shown below



The basic *L*-channel transmultiplexer structure

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Transmultiplexer

- It consists of an *L*-channel synthesis filter bank followed by an *L*-channel analysis filter bank
- It is thus exactly opposite to that of the *L*-channel QMF bank studied earlier

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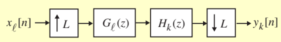
Transmultiplexer

- To determine the input-output relation of the transmultiplexer, consider one typical path from the k-th input to the ℓ–th output as shown below



The k, ℓ-th path

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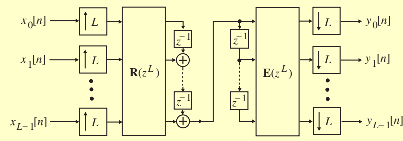
Transmultiplexer

- The polyphase representation of the basic L-channel transmultplexer is shown below



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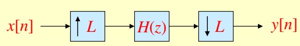
A Multirate Identity

- Consider the structure shown below:



- An L-term Type I polyphase decomposition of $H(z)$ is given by

$$H(z) = \sum_{k=0}^{L-1} z^{-k} E_k(z^L)$$

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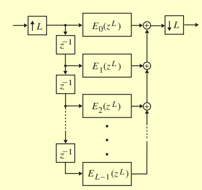
A Multirate Identity

- An equivalent representation of the multirate structure of the previous slide obtained by realizing $H(z)$ in its Type I polyphase form is shown below



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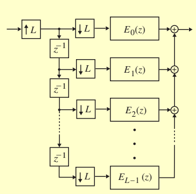
A Multirate Identity

- By moving the down-sampler through the system and invoking the cascade equivalence #1 we arrive at



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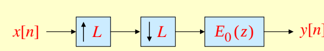
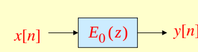
A Multirate Identity

- The structure on the previous slide reduces to the one shown below



- The above finally reduces to the simplified equivalent structure shown below



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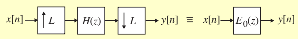
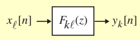
Transmultiplexer

- Invoking the identity



where $E_0(z)$ is the zeroth polyphase term of $H(z)$, the equivalent representation of the (k,ℓ)-th path is equivalent to the structure shown below



where $F_{k\ell}(z)$ is the zeroth polyphase term of $H_k(z)G_\ell(z)$

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Transmultiplexer

- The input-output relation of the transmultiplexer is therefore given by

$$Y_k(z) = \sum_{\ell=0}^{L-1} F_{k\ell}(z) X_\ell(z), \qquad 0 \le k \le L-1$$

- Denoting

$$\mathbf{Y}(z) = [Y_0(z) \quad Y_1(z) \quad \cdots \quad Y_{L-1}(z)]^T$$

$$\mathbf{X}(z) = [X_0(z) \quad X_1(z) \quad \cdots \quad X_{L-1}(z)]^T$$

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Transmultiplexer

we can rewrite the input-output relation as

$$\mathbf{Y}(z) = \mathbf{F}(z)\mathbf{X}(z)$$

where $\mathbf{F}(z)$ is an $L \times L$ matrix whose (k,ℓ)-th element is given by $F_{k\ell}(z)$:

$$\mathbf{F}(z) = \begin{bmatrix} F_{00}(z) & F_{01}(z) & \cdots & F_{0,L-1}(z) \\ F_{10}(z) & F_{11}(z) & \cdots & F_{1,L-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ F_{L-1,0}(z) & F_{L-1,1}(z) & \cdots & F_{L-1,L-1}(z) \end{bmatrix}$$

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Transmultiplexer

- **Design Objective** – To ensure that $y_k[n]$ is a reasonable replica of $x_k[n]$
- If $y_k[n]$ contains contributions from $x_r[n]$ with $r \ne k$, then there is cross-talk between the k-th and r-th channels

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Transmultiplexer

- It follows from

$$\mathbf{Y}(z) = \mathbf{F}(z)\mathbf{X}(z)$$

that cross-talk is totally absent if $\mathbf{F}(z)$ is a diagonal matrix, in which case the above equation reduces to

$$Y_k(z) = F_{kk}(z) X_k(z), \qquad 0 \le k \le L-1$$

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Transmultiplexer

- As in the case of the QMF bank, three types of multiplexer system can be defined

(1) Phase-preserving system if $F_{kk}(z)$ is a linear-phase transfer function for all values of k

(2) Magnitude-preserving system if $F_{kk}(z)$ is an allpass function for all values of k

and

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Transmultiplexer

- (3) Perfect reconstruction system if
  $$F_{kk}(z) = \alpha_k z^{-n_k}, \qquad 0 \le k \le L-1$$
  where n_k is an integer and α_k is a nonzero constant
- For a perfect reconstruction system
  $$y_k[n] = \alpha_k x_k[n]$$

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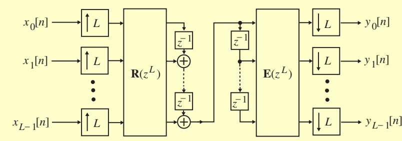
Transmultiplexer

- The perfect reconstruction condition can also be derived in terms of the polyphase components of the synthesis and analysis filter banks of the transmultiplexer shown below



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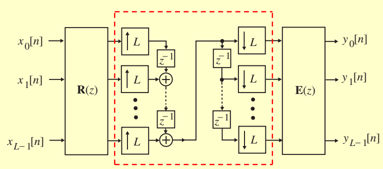
Transmultiplexer

- Using the cascade equivalences we arrive at the equivalent computationally efficient realization shown below



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Transmultiplexer

- Note that the structure inside the dotted box of the figure in the previous slide is a special case of the basic L-channel transmultiplexer structure shown on slide 53 where
  $$G_\ell(z) = z^{-(L-1-\ell)}, \qquad 0 \le \ell, k \le L-1$$
  $$H_k(z) = z^{-k},$$

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Transmultiplexer

- **Note:**

  The zeroth polyphase component of $H_{\ell+1}(z)G_\ell(z)$ is z^{-1} for $0 \le \ell \le L-2$

  The zeroth polyphase component of $H_0(z)G_{L-1}(z)$ is 1

  The zeroth polyphase component of $H_k(z)G_\ell(z)$ is 0 for all other cases

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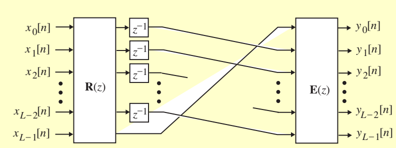
Transmultiplexer

- Thus, a simplified equivalent representation of the structure of slide 69 ia as shown below



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Transmultiplexer

- The transfer matrix characterizing the transmultiplexer is thus given by

$$\mathbf{F}(z) = \mathbf{E}(z)\begin{bmatrix} \mathbf{0} & 1 \\ z^{-1}\mathbf{I}_{L-1} & \mathbf{0} \end{bmatrix}\mathbf{R}(z)$$

where $\mathbf{I}_{L-1}$ is an $(L-1)\times(L-1)$ identity matrix

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Transmultiplexer

- For a perfect reconstruction system, it is sufficient to ensure that

$$\mathbf{F}(z) = d\, z^{-n_0}\mathbf{I}_L,$$

where n_0 is a positive integer

- The condition for perfect reconstruction in terms of the polyphase components is thus

$$\mathbf{R}(z)\mathbf{E}(z) = d\, z^{-m_0}\begin{bmatrix} \mathbf{0} & \mathbf{I}_{L-1} \\ z^{-1} & 0 \end{bmatrix}$$

where m_0 is a suitable positive integer

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Transmultiplexer

- It is possible to develop a perfect reconstruction transmultiplexer from a perfect reconstruction QMF bank with analysis filters $H_\ell(z)$ and synthesis filters $G_\ell(z)$, with a distortion transfer function given by

$$T(z) = d\, z^{-K}$$

where d is a nonzero constant and K is a positive integer

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Transmultiplexer

- To this end, we choose $H_\ell(z)$ as the analysis filters and $z^{-R}G_\ell(z)$ as the synthesis filters of the transmultiplexer, where R is a positive integer less than L such that $R + K$ is a multiple of L

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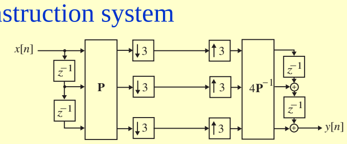
Design Example

- **Example** – The 3-channel QMF bank shown below is by construction a perfect reconstruction system



$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

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Design Example

- Here $\mathbf{E}(z^3) = \mathbf{P}$ and $\mathbf{R}(z^3) = d\,\mathbf{P}^{-1}$
- Hence, $\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}$

resulting in the analysis filters

$$H_0(z) = 1 + z^{-1} + z^{-2}$$
$$H_1(z) = 1 - z^{-1} + z^{-2}$$
$$H_2(z) = 1 - z^{-2}$$

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Design Example

- For $d = 4$, we have

$$d\,\mathbf{P}^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -2 & 0 \\ 1 & 1 & -2 \end{bmatrix}$$

- Hence,

$$\begin{bmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \end{bmatrix} = \begin{bmatrix} z^{-2} & z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & -2 & 0 \\ 1 & 2 & -2 \end{bmatrix}$$

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Design Example

- The synthesis filters of the QMF bank are therefore given by

$$G_0(z) = 1 + 2z^{-1} + z^{-2}$$
$$G_1(z) = 1 - 2z^{-1} + z^{-2}$$
$$G_2(z) = -2 + 2z^{-2}$$

- Here, $d = 4$ and $K = 2$
- We thus choose $R = 1$ so that $R + K = 3$

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Design Example

- The synthesis filters of the transmultiplexer are thus given by

$$z^{-1}G_\ell(z)$$

- We examine the products $z^{-1}G_\ell(z)H_k(z)$, for $0 \le \ell, k \le 2$ and determine their zeroth polyphase components

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Design Example

- Thus, $z^{-1}G_0(z)H_0(z)$

$$= z^{-1} + 3z^{-2} + 4z^{-3} + 3z^{-4} + z^{-5}$$
$$= 4z^{-3} + z^{-1}(1 + 3z^{-3}) + z^{-2}(3 + z^{-3})$$

whose zeroth polyphase component is $4z^{-1}$, and hence,

$$y_0[n] = 4x_0[n-1]$$

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Design Example

- Likewise, $z^{-1}G_1(z)H_1(z)$

$$= z^{-1} - 3z^{-2} + 4z^{-3} - 3z^{-4} + z^{-5}$$
$$= 4z^{-3} + z^{-1}(1 - 3z^{-3}) + z^{-2}(-3 + z^{-3})$$

whose zeroth polyphase component is $4z^{-1}$, and hence,

$$y_1[n] = 4x_1[n-1]$$

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Design Example

- Similarly, $z^{-1}G_2(z)H_2(z)$

$$= -2z^{-1} + 4z^{-3} - 2z^{-5}$$
$$= 4z^{-3} + z^{-1}(-2) + z^{-2}(-2z^{-3})$$

whose zeroth polyphase component is $4z^{-1}$, and hence,

$$y_2[n] = 4x_2[n-1]$$

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Design Example

- Next, consider $z^{-1}G_0(z)H_1(z)$
$$= z^{-1} + z^{-2} + z^{-4} + z^{-5}$$
$$= 0 + z^{-1}(1+z^{-3}) + z^{-2}(1+z^{-3})$$
whose zeroth polyphase component is 0
- In a similar manner, we can show that the zeroth polyphase components for all other products $z^{-1}G_\ell(z)H_k(z)$, with $\ell \neq k$, is 0 indicating a total absence of cross-talks

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TDM-FDM Format Translation

- Typically, 12 digitized speech signals are interpolated by a factor of 12, modulated by SSB modulation, digitally summed, and then converted into an FDM analog signal by D/A conversion
- At the receiving end, the analog signal is converted into a digital signal by A/D conversion

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TDM-FDM Format Translation

- The analog FDM signal is passed through a bank of 12 SSB demodulators whose outputs are then decimated, resulting in the low-frequency speech signals
- The speech signals have a bandwidth of 4 kHz and are sampled at 8-kHz rate
- The FDM signal signal occupies the band 60 kHz to 108 kHz

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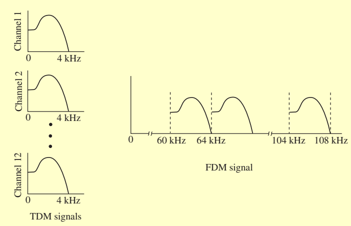
TDM-FDM Format Translation

- Spectrums of TDM and FDM signals



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TDM-FDM Format Translation

- The interpolation and the single-sideband modulation can be performed by up-sampling and appropriate filtering
- Likewise, the single-sideband demodulation and decimation can be implemented by appropriate filtering and down-sampling

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