Perfect Reconstruction Two-Channel FIR Filter Banks

- A perfect reconstruction two-channel FIR filter bank with linear-phase FIR filters can be designed if the power-complementary requirement

$$|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$$

between the two analysis filters $H_0(z)$ and $H_1(z)$ is not imposed

1

Modulation matrix

- To develop the pertinent design equations we observe that the input-output relation of the 2-channel QMF bank

$$Y(z) = \tfrac{1}{2}\{H_0(z)G_0(z) + H_1(z)G_1(z)\}X(z)$$
$$+ \tfrac{1}{2}\{H_0(-z)G_0(z) + H_1(-z)G_1(z)\}X(-z)$$

can be expressed in matrix form as

$$Y(z) = \tfrac{1}{2}[G_0(z) \;\; G_1(z)]\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}\begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

2

Modulation matrix

- From the previous equation we obtain

$$Y(-z) = \tfrac{1}{2}[G_0(-z) \;\; G_1(-z)]\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}\begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

- Combining the two matrix equations we get

$$\begin{bmatrix} Y(z) \\ Y(-z) \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} G_0(z) & G_1(z) \\ G_0(-z) & G_1(-z) \end{bmatrix}\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}\begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$
$$= \tfrac{1}{2}\mathbf{G}^{(m)}(z)[\mathbf{H}^{(m)}(z)]^T\begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

3

Modulation matrix

where

$$\mathbf{H}^{(m)}(z) = \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}$$

is the **analysis modulation matrix** and

$$\mathbf{G}^{(m)}(z) = \begin{bmatrix} G_0(z) & G_1(z) \\ G_0(-z) & G_1(-z) \end{bmatrix}$$

is the **synthesis modulation matrix**

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Perfect Reconstruction Condition

- Now for perfect reconstruction we must have $Y(z) = z^{-\ell}X(z)$ and correspondingly $Y(-z) = (-z)^{-\ell}X(-z)$
- Substituting these relations in the equation

$$\begin{bmatrix} Y(z) \\ Y(-z) \end{bmatrix} = \tfrac{1}{2}\mathbf{G}^{(m)}(z)[\mathbf{H}^{(m)}(z)]^T\begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

we observe that the PR condition is satisfied if $\mathbf{G}^{(m)}(z)[\mathbf{H}^{(m)}(z)]^T = 2\begin{bmatrix} z^{-\ell} & 0 \\ 0 & (-z)^{-\ell} \end{bmatrix}$

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Perfect Reconstruction Two-Channel FIR Filter Banks

- Thus, knowing the analysis filters $H_0(z)$ and $H_1(z)$, the synthesis filters $G_0(z)$ and $G_1(z)$ are determined from

$$\mathbf{G}^{(m)}(z) = 2\begin{bmatrix} z^{-\ell} & 0 \\ 0 & (-z)^{-\ell} \end{bmatrix}\left([\mathbf{H}^{(m)}(z)]^T\right)^{-1}$$

- After some algebra we arrive at

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Perfect Reconstruction Two-Channel FIR Filter Banks

$$G_0(z) = \frac{2z^{-\ell}}{\det[\mathbf{H}^{(m)}(z)]} \cdot H_1(-z)$$

$$G_1(z) = -\frac{2z^{-\ell}}{\det[\mathbf{H}^{(m)}(z)]} \cdot H_0(-z)$$

where

$$\det[\mathbf{H}^{(m)}(z)] = H_0(z)H_1(-z) - H_0(-z)H_1(z)$$

and ℓ is an odd positive integer

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Perfect Reconstruction Two-Channel FIR Filter Banks

- For FIR analysis filters $H_0(z)$ and $H_1(z)$, the synthesis filters $G_0(z)$ and $G_1(z)$ will also be FIR filters if

$$\det[\mathbf{H}^{(m)}(z)] = cz^{-k}$$

where c is a real number and k is a positive integer

- In this case $G_0(z) = \frac{2}{c} z^{-(\ell-k)} H_1(-z)$

$$G_1(z) = -\frac{2}{c} z^{-(\ell-k)} H_0(-z)$$

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Biorthogonal Filter Banks

- For a perfect reconstruction filter bank, the normalized product filter $P(z)$ must be a zero-phase half-band lowpass filter
- $\Rightarrow$ $P(z)$ is a symmetric polynomial of the form

$$P(z) = 1 + p_1(z + z^{-1}) + p_3(z^3 + z^{-3})$$
$$+ p_5(z^5 + z^{-5}) + \cdots$$

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Biorthogonal Filter Banks

- In QMF banks for signal compression application, it is preferable to choose the lowpass filters, $H_0(z)$ and $G_0(z)$, with a maximum number of zeros at $z = -1$, which are also zeros of $P_0(z)$, and hence $P(z)$
- The general form of $P(z)$ is then given by

$$P(z) = (1 + z^{-1})^m (1 + z)^m R(z)$$

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Biorthogonal Filter Banks

where $R(z)$ is a symmetric polynomial, i.e., $R(z) = R(z^{-1})$, with $R(e^{j\omega}) \geq 0$

- This class of half-band filters has been called the binomial or maxflat filter, as they have a frequency response $P(e^{j\omega})$ that is maximally flat at $\omega = 0$ and $\omega = \pi$

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Biorthogonal Filter Banks

- When $R(z)$ is of the form

$$R(z) = r_0 + \sum_{s=1}^{m-1} r_s (z^s + z^{-s})$$

it is said to be of minimal degree and is of practical interest

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Biorthogonal Filter Banks

- Consider the case $m = 1$ and $R(z) = \frac{1}{2}$
- Then $P(z) = \frac{1}{2}(z + 2 + z^{-1})$

$$= \tfrac{1}{2} z(1+z^{-1})(1+z^{-1}) = z^{\ell} H_0(z) G_0(z)$$

- If we choose $H_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$, the lowest order $G_0(z)$ is obtained with $\ell = 1$ and is given by $G_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$

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Biorthogonal Filter Banks

- Next, consider the case $m = 2$
- Then, $R(z)$ is of the form $R(z) = az + b + az^{-1}$
- As a result,

$$P(z) = (1+z^{-1})^2 (1+z)^2 (az + b + az^{-1})$$
$$= az^3 + (4a+b)z^2 + (7a+4b)z + (8a+6b)$$
$$+ (7a+4b)z^{-1} + (4a+b)z^{-2} + az^{-3}$$

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Biorthogonal Filter Banks

- Since even powers of $P(z)$ must be zeros and the coefficient of z^0 must be equal to 1, we have

$$4a + b = 0, \qquad 8a + 6b = 1$$

- Solving the above two equations we get

$$a = -\tfrac{1}{16}, \qquad b = \tfrac{1}{4}$$

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Biorthogonal Filter Banks

- Thus, in this case, we have

$$P(z) = [1 + \tfrac{1}{2}(z + z^{-1})]^2 [1 - \tfrac{1}{4}(z + z^{-1})]$$
$$= \tfrac{1}{16} z^3 (1 + 2z^{-1} + z^{-2})^2 (-1 + 4z^{-1} - z^{-2})$$

- One possible factorization of $P(z)$, with $\ell = 3$, is given by

$$H_0(z) = \tfrac{1}{2}(1 + 2z^{-1} + z^{-2})$$
$$G_0(z) = \tfrac{1}{8}(1 + 2z^{-1} + z^{-2})(-1 + 4z^{-1} - z^{-2})$$

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Biorthogonal Filter Banks

- The corresponding highpass filters are given by

$$H_1(z) = G_0(-z)$$
$$= \tfrac{1}{8}(-1 - 2z^{-1} + 6z^{-2} - 2z^{-3} - z^{-4})$$
$$G_1(z) = -H_0(-z) = -\tfrac{1}{2}(1 - 2z^{-1} + z^{-2})$$

- The length-3 lowpass and the length-5 highpass analysis filters are often referred to as the LeGall 3/5-tap filter pair

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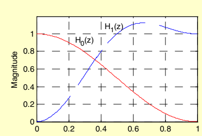
Biorthogonal Filter Banks

- A plot of the magnitude responses of the LeGall 3/5-tap analysis filter pairs are shown below:



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Biorthogonal Filter Banks

- A different set of perfect reconstruction QMF filters, called LeGall 5/3-tap filter pairs, are obtained by interchanging the factors assigned to the two lowpass filters:

$$H_0(z) = \tfrac{1}{8}(-1 + 2z^{-1} + 6z^{-2} + 2z^{-3} - z^{-4})$$
$$G_0(z) = \tfrac{1}{2}(1 + 2z^{-1} + z^{-2})$$
$$H_1(z) = \tfrac{1}{2}(1 - 2z^{-1} + z^{-2})$$
$$G_1(z) = \tfrac{1}{8}(1 + 2z^{-1} - 6z^{-2} + 2z^{-3} + z^{-4})$$

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Biorthogonal Filter Banks

- Another possible factorization of $P(z)$, with $\ell = 3$, leads to the lowpass filters

$$H_0(z) = \tfrac{1}{8}(1 + 3z^{-1} + 3z^{-2} + z^{-3})$$
$$G_0(z) = \tfrac{1}{2}(-1 + 3z^{-1} + 3z^{-2} - z^{-3})$$

- The corresponding highpass filters are given by

$$H_1(z) = \tfrac{1}{8}(-1 - 3z^{-1} + 3z^{-2} + z^{-3})$$
$$G_1(z) = \tfrac{1}{2}(-1 + 3z^{-1} - 3z^{-2} + z^{-3})$$

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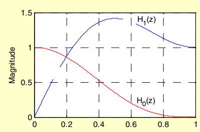
Biorthogonal Filter Banks

- The analysis filters of the previous slide are more commonly known as the Daubechies 4/4-tap filter pair
- A plot of the magnitude responses of these filters are shown below



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Biorthogonal Filter Banks

- It can be shown that the above choices for the analysis filters do satisfy the condition

$$\det[\mathbf{H}^{(m)}(z)] = cz^{-k}$$

on the determinant of the analysis modulation matrix

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Biorthogonal Filter Banks

- For example, for the Haar filters we have

$$\det[\mathbf{H}^{(m)}(z)] = H_0(z)H_1(-z) - H_0(-z)H_1(z)$$
$$= P_0(z) - P_0(-z)$$
$$= \tfrac{1}{2}(1 + z^{-1})^2 - \tfrac{1}{2}(1 - z^{-1})^2 = 2z^{-1}$$

which is seen to satisfy the condition with $c = 2$ and $k = 1$

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Biorthogonal Filter Banks

- Likewise, for the LeGall 3/5-tap filter pair, we have

$$\det[\mathbf{H}^{(m)}(z)]$$
$$= \tfrac{1}{16}(1 + 2z^{-1} + z^{-2})^2(-1 + 4z^{-1} - z^{-2})$$
$$- \tfrac{1}{16}(1 - 2z^{-1} + z^{-2})^2(-1 - 4z^{-1} - z^{-2})$$
$$= 2z^{-3}$$

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Biorthogonal Filter Banks

- In many applications, it is desirable to use a QMF bank in which the outputs of the analysis filters are represented as an orthonormal transformation of the input signal, while the reconstruction process is represented by a transformation that is simply the transpose of the analysis filtering transform matrix

Biorthogonal Filter Banks

- However, the subband filtering process described so far cannot be represented as an orthonormal transformation of the input signal
- QMF banks obtained using perfect reconstruction linear-phase FIR filters and represented by nonorthonormal transformation matrices are often referred to as the biorthogonal filter banks

Orthogonal Filter Banks

- These filter banks are designed using a different scheme for the factorization of the lowpass product filter $P(z)$ of order $2N$, which results in FIR analysis and synthesis filters that are no longer linear-phase
- Now, $P(z)$ being a symmetric polynomial of the form

  $$P(z) = 1 + p_1(z + z^{-1}) + p_3(z^3 + z^{-3}) + \cdots,$$

  it has factors of the form $(\alpha z + 1)(1 + \alpha z^{-1})$

Orthogonal Filter Banks

- We can assign the subfactor $(1 + \alpha z^{-1})$ to $H_0(z)$ and the subfactor $z^{-1}(\alpha z + 1)$ to $G_0(z)$
- This process is continued with respect to each factor of $P(z)$, leading to $H_0(z)$ and its mirror-image transfer function

  $$G_0(z) = z^{-N} H_0(z^{-1})$$

Orthogonal Filter Banks

- For real-coefficient filters, the above relation implies that

  $$G_0(e^{j\omega}) = H_0(e^{-j\omega})$$

- Thus, the magnitude responses of the two lowpass filters are the same, while the phase responses are opposite of each other

Orthogonal Filter Banks

- As in the case of the biorthogonal filter banks, it is desirable to have filters with as many zeros as possible at $z = -1$, that is, to have a maximum number of zeros of $P(z)$ at $z = -1$
- We can thus choose $P(z)$ to be of the form

  $$P(z) = (1 + z^{-1})^m (1 + z)^m R(z)$$

Orthogonal Filter Banks

- We can thus assign, for example, all factors of $P(z)$ having zeros inside the unit circle to $H_0(z)$ and assign all factors of $P(z)$ having zeros outside the unit circle to $G_0(z)$
- Moreover, zeros of $P(z)$ on the unit circle are of even multiplicity, and hence, half of these zeros are assigned to $H_0(z)$ and the other half to $G_0(z)$

Orthogonal Filter Banks

- This assignment makes $G_0(z)$ the mirror-image of $H_0(z)$
- In addition, $H_0(z)$ is a minimum-phase filter, whereas, $G_0(z)$ is a maximum-phase filter
- As an example, consider the factorization of

$$P(z) = \tfrac{1}{16} z^3 (1 + 2z^{-1} + z^{-2})^2 (-1 + 4z^{-1} - z^{-2})$$

Orthogonal Filter Banks

- Note that the factor $(1 - 4z^{-1} + z^{-2})$ has a zero inside the unit circle at $z = 2 - \sqrt{3}$ and another zero outside the unit circle at $z = 2 + \sqrt{3}$
- The minimum-phase spectral factor of $P(z)$ thus yields the lowpass analysis filter

$$H_0(z) = \tfrac{1}{4(\sqrt{3}-1)} (1 + z^{-1})^2 \left(1 - (2 - \sqrt{3}) z^{-1}\right)$$
$$= 0.3415 + 0.5915 z^{-1} + 0.1585 z^{-2} - 0.0915 z^{-3}$$

Orthogonal Filter Banks

- The maximum-phase spectral factor yields the lowpass synthesis filter

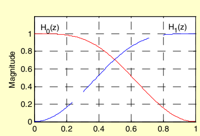
$$G_0(z) = z^{-3} H_0(z^{-1})$$
$$= -0.0915 + 0.1585 z^{-1} + 0.5915 z^{-2} + 0.3415 z^{-3}$$

- The two highpass filters are given by

$$H_1(z) = G_0(-z)$$
$$= -0.0915 - 0.1585 z^{-1} + 0.5915 z^{-2} - 0.3415 z^{-3}$$
$$G_1(z) = -H_0(-z)$$
$$= -0.3415 + 0.5915 z^{-1} - 0.1585 z^{-2} - 0.0915 z^{-3}$$

Orthogonal Filter Banks

- The magnitude responses of the two analys filters are shown below



Orthogonal Filter Banks

- A general method for the design of a lowpass zero-phase half-band filter $P(z) = H_0(z) H_0(z^{-1})$ of order $2N$ with N odd is outlined next
- Let $H_0(z)$ be an FIR filter of odd order N satisfying the power-symmetric condition

$$H_0(z) H_0(z^{-1}) + H_0(-z) H_0(-z^{-1}) = 1$$

- Choose $H_1(z) = -z^{-N} H_0(-z^{-1})$

Orthogonal Filter Banks

- Then, the determinant of the analysis modulation matrix is given by

$$\det[\mathbf{H}^{(m)}(z)] = H_0(z)H_1(-z) - H_0(-z)H_1(z)$$
$$= z^{-N}\left(H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1})\right)$$
$$= z^{-N}$$

- Thus, the perfect reconstruction condition is satisfied with $c = 1$ and $k = N$

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Orthogonal Filter Banks

- For the synthesis filters we have for

$$G_0(z) = z^{-N}H_0(z^{-1}), \quad G_1(z) = z^{-N}H_1(z^{-1})$$

- Note: If $H_0(z)$ is a causal FIR filter, then the other 3 filters are also causal FIR filters
- Moreover, $|G_i(e^{j\omega})| = |H_i(-e^{j\omega})|, \; i = 1, 2$
- In addition, $|H_1(e^{j\omega})| = |H_0(-e^{j\omega})|$

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Orthogonal Filter Banks

- $\Rightarrow$ For a real coefficient transfer function, if $H_0(z)$ is a lowpass filter, then $H_1(z)$ is a highpass filter
- A perfect reconstruction power-symmetric filter bank is also called an orthogonal filter bank
- The filter bank design problem thus reduces to the design of a power-symmetric lowpass filter

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Orthogonal Filter Banks

- To this end, we can design an even order $P_0(z) = H_0(z)H_0(z^{-1})$ whose spectral factorization yields $H_0(z)$
- Now, the power-symmetric condition implies that $P_0(z)$ be a zero-phase half-band lowpass filter with a non-negative frequency response $P_0(e^{j\omega})$
- Such a filter can be designed using the minimum-phase FIR filter design method

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Orthogonal Filter Banks

- Several comments are in order here
- 1) The order of the half-band filter $P_0(z)$ is of the form $4K+2$, where K is a positive integer
- $\Rightarrow$ The order of $H_0(z)$ is $N = 2K+1$, which is odd as desired

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Orthogonal Filter Banks

- 2) Zeros of $P_0(z)$ appear with mirror-image symmetry in the z-plane, with the zeros on the unit circle being of multiplicity 2
- Any appropriate half of these zeros can be grouped to form the spectral factor $H_0(z)$
- For example, a minimum-phase $H_0(z)$ can be formed by grouping all zeros inside the unit circle with half of the zeros on the unit circle

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Orthogonal Filter Banks

- However, it is not possible to form a spectral factor with a linear phase
- 3) The stopband edge frequency is the same for $P_0(z)$ and $H_0(z)$
- If the desired minimum stopband attenuation of $H_0(z)$ is α_s dB, then the minimum stopband attenuation of $P_0(z)$ is $2\alpha_s + 6.02$ dB

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Orthogonal Filter Banks

- The steps for the design of a real-coefficient power-symmetric lowpass filter $H_0(z)$ are:
- (1) Design a zero-phase real-coefficient FIR half-band lowpass filter $Q(z)$ of order $2N$ with N an odd positive integer:

$$Q(z) = \sum_{n=-N}^{N} q[n] z^{-n}$$

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Orthogonal Filter Banks

- (2) Let δ denote the peak stopband ripple of $Q(e^{j\omega})$
- Define $P_0(z) = Q(z) + \delta$ which guarantees that $P_0(e^{j\omega}) \geq 0$ for all ω
- Note: If $q[n]$ denotes the impulse response of $Q(z)$, then the impulse response $p_0[n]$ of $P_0(z)$ is given by

$$p_0[n] = \begin{cases} q[n]+\delta, & \text{for } n=0 \\ q[n], & \text{for } n \neq 0 \end{cases}$$

- (3) Determine the spectral factor $H_0(z)$ of $P_0(z)$

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Orthogonal Filter Banks

- Example - Design a lowpass real-coefficient power-symmetric filter $H_0(z)$ with the following specifications: $\omega_s = 0.63\pi$, and $\alpha_s = 13$ dB
- The specifications of the corresponding zero-phase half-band filter $P_0(z)$ are therefore: $\omega_s = 0.63\pi$ and $\alpha_s = 40$ dB

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Orthogonal Filter Banks

- The desired stopband ripple is thus $\delta_s = 0.01$ which is also the passband ripple
- The passband edge is at

$$\omega_p = \pi - 0.63\pi = 0.37\pi$$

- Using the function `remezord` we first estimate the order of $P_0(z)$ and then using the function `remez` design $Q(z)$

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Orthogonal Filter Banks

- The order of $F(z)$ is found to be 14 implying that the order of $H_0(z)$ is 7 which is odd as desired
- To determine the coefficients of $F(z)$ we add `err` (the maximum stopband ripple) to the central coefficient $q[7]$
- Next, using the function `roots` we determine the roots of $F(z)$ which should theretically exhibit mirror-image symmetry with double roots on the unit circle

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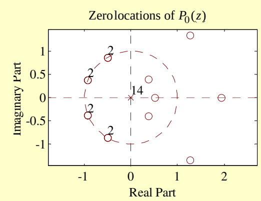
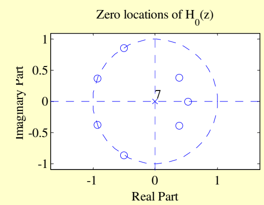
Orthogonal Filter Banks

- However, the algorithm s numerically quite sensitive and it is found that a slightly larger value than `err` should be added to ensure double zeros of $P_0(z)$ on the unit circle
- Choosing the roots inside the unit circle along with one set of unit circle roots we get the minimum-phase spectral factor $H_0(z)$

Orthogonal Filter Banks

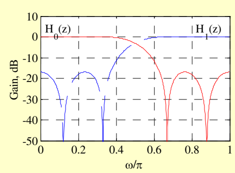
- The zero locations of $P_0(z)$ and $H_0(z)$ are shown below





Orthogonal Filter Banks

- The gain responses of the two analysis filters are shown below



Orthogonal Filter Banks

- Separate realizations of the two filters $H_0(z)$ and $H_1(z)$ would require $2(N+1)$ multipliers and $2N$ two-input adders
- However, a computationally efficient realization requiring $N+1$ multipliers and $2N$ two-input adders can be developed by exploiting the relation

$$H_1(z) = z^{-N} H_0(-z^{-1})$$

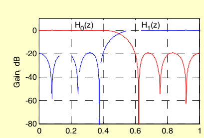
Orthogonal Filter Banks

- The M-file `firpr2chfb` can be used to design a perfect reconstruction two-channel orthogonal filter bank as shown next
- Filter order is 11 and normalized angular passband edge frequency of the lowpass filters is 0.4
- The code fragments used is

```
[h0,h1,g0,g1] = firpr2chfb(11,0.4);
```


Orthogonal Filter Banks

- A plot of the gain responses of the two analysis filters is shown below



Paraunitary System

- A p-input, q-output LTI discrete-time system with a transfer matrix $\mathbf{T}_{pq}(z)$ is called a paraunitary system if $\mathbf{T}_{pq}(z)$ is a paraunitary matrix, i.e.,

$$\tilde{\mathbf{T}}_{pq}(z)\mathbf{T}_{pq}(z) = c\mathbf{I}_p$$

- Note: $\tilde{\mathbf{T}}_{pq}(z)$ is the paraconjugate of $\mathbf{T}_{pq}(z)$ given by the transpose of $\mathbf{T}_{pq}(z^{-1})$ with each coefficient replaced by its conjugate
- $\mathbf{I}_p$ is an $p \times p$ identity matrix, c is a real constant

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Paraunitary Filter Banks

- A causal, stable paraunitary system is also a lossless system
- It can be shown that the modulation matrix

$$\mathbf{H}^{(m)}(z) = \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}$$

of a power-symmetric filter bank is a paraunitary matrix

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Paraunitary Filter Banks

- Hence, a power-symmetric filter bank has also been referred to as a paraunitary filter bank
- The cascade of two paraunitary systems with transfer matrices $\mathbf{T}_{pq}^{(1)}(z)$ and $\mathbf{T}_{qr}^{(2)}(z)$ is also paraunitary
- The above property can be utilized in designing a paraunitary filter bank without resorting to spectral factorization

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Power-Symmetric FIR Cascaded Lattice Structure

- Consider a real-coefficient FIR transfer function $H_N(z)$ of order N satisfying the power-symmetric condition

$$H_N(z)H_N(z^{-1}) + H_N(-z)H_N(-z^{-1}) = K_N$$

- We shall show now that $H_N(z)$ can be realized in the form of a cascaded lattice structure as shown on the next slide

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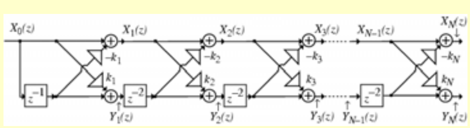
Power-Symmetric FIR Cascaded Lattice Structure



- Define

$$H_i(z) = \frac{X_i(z)}{X_0(z)}, \quad G_i(z) = \frac{Y_i(z)}{X_0(z)}$$

$$1 \le i \le N$$

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Power-Symmetric FIR Cascaded Lattice Structure

- From the figure we observe that

$$X_1(z) = X_0(z) + k_1 z^{-1} X_0(z)$$

$$Y_1(z) = -k_1 X_0(z) + z^{-1} X_0(z)$$

- Therefore,

$$H_1(z) = 1 + k_1 z^{-1}, \quad G_1(z) = -k_1 + z^{-1}$$

- It can be easily shown that

$$G_1(z) = z^{-1} H_1(-z^{-1})$$

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Power-Symmetric FIR Cascaded Lattice Structure

- Next from the figure it follows that

$$H_i(z) = H_{i-2}(z) + k_i z^{-2} G_{i-2}(z)$$
$$G_i(z) = -k_i H_{i-2}(z) + z^{-2} G_{i-2}(z)$$

- It can easily be shown that

$$G_i(z) = z^{-i} H_i(-z^{-1})$$

provided

$$G_{i-2}(z) = z^{-(i-2)} H_{i-2}(-z^{-1})$$

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Power-Symmetric FIR Cascaded Lattice Structure

- We have shown that $G_i(z) = z^{-i} H_i(-z^{-1})$ holds for i = 1
- Hence the above relation holds for all odd integer values of i
- $\Rightarrow$ N must be an odd integer
- It is a simple exercise to show that both $H_i(z)$ and $G_i(z)$ satisfy the power-symmetry condition $H_i(z)H_i(z^{-1}) + H_i(-z)H_i(-z^{-1}) = K_i$

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Power-Symmetric FIR Cascaded Lattice Structure

- In addition, $H_i(z)$ and $G_i(z)$ are power-complementary, i.e.,

$$(1 + k_i^2) z^{-2} G_{i-2}(z) = k_i H_i(z) + G_i(z)$$

- To develop the synthesis equation we express $H_{i-2}(z)$ and $G_{i-2}(z)$ in terms of $H_i(z)$ and $G_i(z)$:

$$(1 + k_i^2) H_{i-2}(z) = H_i(z) - k_i G_i(z)$$
$$(1 + k_i^2) z^{-2} G_{i-2}(z) = k_i H_i(z) + G_i(z)$$

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Power-Symmetric FIR Cascaded Lattice Structure

- Note: At the i-th step, the coefficient k_i is chosen to eliminate the coefficient of z^{-i}, the highest power of z^{-1} in $H_i(z) - k_i G_i(z)$
- For this choice of k_i the coefficient of also vanishes making $H_{i-2}(z)$ a polynomial of degree $i-2$
- The synthesis process begins with $i = N$ and compute $G_N(z)$ using $G_N(z) = z^{-N} H_N(-z^{-1})$

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Power-Symmetric FIR Cascaded Lattice Structure

- Next, the transfer functions $H_{N-2}(z)$ and $G_{N-2}(z)$ are determined using the synthesis equations

$$(1 + k_i^2) H_{i-2}(z) = H_i(z) - k_i G_i(z)$$
$$(1 + k_i^2) z^{-2} G_{i-2}(z) = k_i H_i(z) + G_i(z)$$

- This process is repeated until all coefficients of the lattice have been determined

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Power-Symmetric FIR Cascaded Lattice Structure

- Example - Consider

$$H_5(z) = 1 + 0.3z^{-1} + 0.2z^{-2} - 0.376z^{-3} - 0.06z^{-4} + 0.2z^{-5}$$

- It can be easily verified that $H_5(z)$ satisfies the power-symmetric condition
- Next we form

$$G_5(z) = z^{-5} H_5(-z^{-1}) = -0.2 - 0.06z^{-1} + 0.376z^{-2} + 0.2z^{-3} - 0.3z^{-4} + z^{-5}$$

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Power-Symmetric FIR Cascaded Lattice Structure

- To determine $H_5(z)$ we first form

$$H_5(z)-k_5G_5(z)=(1+0.2k_5)+(0.3+0.06k_5)z^{-1}$$
$$+(0.2-0.376k_5)z^{-2}+(-0.376-0.2k_5)z^{-3}$$
$$+(-0.06+0.3k_5)z^{-4}+(0.2-k_5)z^{-5}$$

- To cancel the coefficient of z^{-5} in the above we choose

$$k_5=0.2$$

Power-Symmetric FIR Cascaded Lattice Structure

- Then $H_3(z)=\frac{1}{1-k_5^2}[H_5(z)-k_5G_5(z)]$

$$=\tfrac{1}{1.04}(1.04+0.312z^{-1}+0.1248z^{-2}-0.416z^{-3})$$

- We next form

$$G_3(z)=z^{-3}H_3(-z^{-1})=0.4+0.12z^{-1}-0.3z^{-2}+z^{-3}$$

- Continuing the above process we get

$$k_3=-0.4,\quad k_1=0.3$$

Power-Symmetric FIR Banks

- Using the method outlined for the realization of a power-symmetric transfer function, we can develop a cascaded lattice realization of the 2-channel paraunitary QMF bank
- Three important properties of the QMF lattice structure are structurally induced

Power-Symmetric FIR Banks

- (1) The QMF lattice guarantees perfect reconstruction independent of the lattice parameters
- (2) It exhibits very small coefficient sensitivity to lattice parameters as each stage remains lossless under coefficient quantization
- (3) Computational complexity is about one-half that of any other realization as it requires $(N+1)/2$ total number of multipliers for an order-N filter

Power-Symmetric FIR Banks

- Example - Consider the analysis filter of the previous example:

$$H_7(z)=0.3231+0.51935z^{-1}+0.30134z^{-2}$$
$$-0.0781z^{-3}-0.13767z^{-4}+0.321z^{-5}$$
$$+0.079z^{-6}-0.049z^{-7}$$

- We place a multiplier $h[0]=0.3231$ at the input and synthesize a cascade lattice structure for the normalized transfer function $H_7(z)/h[0]$

Power-Symmetric FIR Banks

- The lattice coefficients obtained for the normalized analysis transfer function are:

$$k_7=-0.15165,\quad k_5=0.2354,$$
$$k_3=-0.48393,\quad k_1=1.61$$

- Note: Because of the numerical problem, the coefficients of the spectral factor obtained in the previous example are not very accurate

Power-Symmetric FIR Banks

- As a result, the coefficients of $z^{-(i-1)}$ of the transfer function $H_{i-2}(z)$ generated from the transfer function $H_i(z)$ using the relation
$$H_{i-2}(z) = \frac{1}{1+k_i^2}[H_i(z) - k_i G_i(z)]$$
is not exactly zero, and has been set to zero at each iteration

Power-Symmetric FIR Banks

- Two interesting properties of the cascaded lattice QMF bank can be seen by examining its multiplier coefficient values
- (1) Signs of coefficients alternate between stages
- (2) The values of the coefficients $\{k_i\}$ decrease with increasing i

Power-Symmetric FIR Banks

- The QMF lattice structure can be used directly to design the power-symmetric analysis filter $H_0(z)$ using an iterative computer-aided optimization technique
- Goal: Determine the lattice parameters k_i by minimizing the energy in the stopband of $H_0(z)$

Power-Symmetric FIR Banks

- The pertinent objective function is given by
$$\phi = \int_{\omega_s}^{\pi} \left| H_0(e^{j\omega}) \right|^2 d\omega$$
- Note: The power-symmetric property ensures good passband response

Orthogonal Filter Banks