Quadrature-Mirror Filter Bank

- In many applications, a discrete-time signal $x[n]$ is split into a number of subband signals $\{v_k[n]\}$ by means of an analysis filter bank
- The subband signals are then processed
- Finally, the processed subband signals are combined by a synthesis filter bank resulting in an output signal $y[n]$

Quadrature-Mirror Filter Bank

- If the subband signals $\{v_k[n]\}$ are bandlimited to frequency ranges much smaller than that of the original input signal $x[n]$, they can be down-sampled before processing
- Because of the lower sampling rate, the processing of the down-sampled signals can be carried out more efficiently

Quadrature-Mirror Filter Bank

- After processing, these signals are then up-sampled before being combined by the synthesis filter bank into a higher-rate signal
- The combined structure is called a quadrature-mirror filter (QMF) bank

Quadrature-Mirror Filter Bank

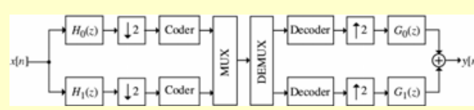
- If the down-sampling and up-sampling factors are equal to or greater than the number of bands of the filter bank, then the output $y[n]$ can be made to retain some or all of the characteristics of the input signal $x[n]$ by choosing appropriately the filters in the structure

Quadrature-Mirror Filter Bank

- If the up-sampling and down-sampling factors are equal to the number of bands, then the structure is called a critically sampled filter bank
- The most common application of this scheme is in the efficient coding of a signal $x[n]$

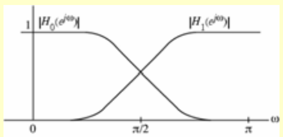
Two-Channel QMF Bank

- Figure below shows the basic two-channel QMF bank-based subband codec (coder/decoder)



Two-Channel QMF Bank

- The analysis filters $H_0(z)$ and $H_1(z)$ have typically a lowpass and highpass frequency responses, respectively, with a cutoff at $\pi/2$



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Two-Channel QMF Bank

- Each down-sampled subband signal is encoded by exploiting the special spectral properties of the signal, such as energy levels and perceptual importance
- It follows from the figure that the sampling rates of the output $y[n]$ and the input $x[n]$ are the same

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Two-Channel QMF Bank

- The analysis and the synthesis filters are chosen so as to ensure that the reconstructed output $y[n]$ is a reasonably close replica of the input $x[n]$
- Moreover, they are also designed to provide good frequency selectivity in order to ensure that the sum of the power of the subband signals is reasonably close to the input signal power

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Two-Channel QMF Bank

- In practice, various errors are generated in this scheme
- In addition to the coding error and errors caused by transmission of the coded signals through the channel, the QMF bank itself introduces several errors due to the sampling rate alterations and imperfect filters
- We ignore the coding and the channel errors

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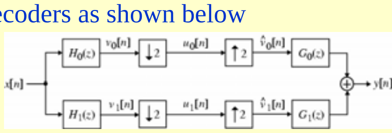
Two-Channel QMF Bank

- We investigate only the errors caused by the sampling rate alterations and their effects on the performance of the system
- To this end, we consider the QMF bank structure without the coders and the decoders as shown below



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Analysis of the Two-Channel QMF Bank

- Making use of the input-output relations of the down-sampler and the up-sampler in the z-domain we arrive at

$$V_k(z) = H_k(z)X(z),$$

$$U_k(z) = \tfrac{1}{2}\{V_k(z^{1/2}) + V_k(-z^{1/2})\}, \quad k = 0, 1$$

$$\hat{V}_k(z) = U_k(z^2)$$

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Analysis of the Two-Channel QMF Bank

- From the first and the last equations we obtain after some algebra

$$\hat{V}_k(z) = \tfrac{1}{2}\{V_k(z) + V_k(-z)\}$$
$$= \tfrac{1}{2}\{H_k(z)X(z) + H_k(-z)X(-z)\}$$

- The reconstructed output of the filter bank is given by

$$Y(z) = G_0(z)\hat{V}_0(z) + G_1(z)\hat{V}_1(z)$$

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Analysis of the Two-Channel QMF Bank

- From the two equations of the previous slide we arrive at

$$Y(z) = \tfrac{1}{2}\{H_0(z)G_0(z) + H_1(z)G_1(z)\}X(z)$$
$$+ \tfrac{1}{2}\{H_0(-z)G_0(z) + H_1(-z)G_1(z)\}X(-z)$$

- The second term in the above equation is due to the aliasing caused by sampling rate alteration

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Analysis of the Two-Channel QMF Bank

- The input-output equation of the filter bank can be compactly written as

$$Y(z) = T(z)X(z) + A(z)X(-z)$$

where $T(z)$, called the distortion transfer function, is given by

$$T(z) = \tfrac{1}{2}\{H_0(z)G_0(z) + H_1(z)G_1(z)\}$$

and

$$A(z) = \tfrac{1}{2}\{H_0(-z)G_0(z) + H_1(-z)G_1(z)\}$$

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Two-Channel QMF Bank

- Since the up-sampler and the down-sampler are linear time-varying components, in general, the 2-channel QMF structure is a linear time-varying system
- It can be shown that the 2-channel QMF structure has a period of 2
- However, it is possible to choose the analysis and synthesis filters such that the aliasing effect is canceled resulting in a time-invariant operation

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Aliasing Cancellation Condition

- The aliasinging is cancelled if $A(z) = 0$
- The aliasing cancellation condition is thus given by

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$$

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Alias-Free Filter Bank

- If the above relations hold, then the QMF system is time-invariant with an input-output relation given by

$$Y(z) = T(z)X(z)$$

where

$$T(z) = \tfrac{1}{2}\{H_0(z)H_1(-z) + H_1(z)H_0(-z)\}$$

- On the unit circle, we have

$$Y(e^{j\omega}) = T(e^{j\omega})X(e^{j\omega}) = |T(e^{j\omega})|\,e^{j\phi(\omega)}X(e^{j\omega})$$

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Alias-Free Filter Bank

- If $T(z)$ is an allpass function, i.e., $|T(e^{j\omega})| = d$ with $d \neq 0$ then
$$|Y(e^{j\omega})| = d\,|X(e^{j\omega})|$$
indicating that the output of the QMF bank has the same magnitude response as that of the input (scaled by d) but exhibits phase distortion
- The filter bank is said to be magnitude preserving

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Alias-Free Filter Bank

- If $T(z)$ has linear phase, i.e.,
$$\arg\{T(e^{j\omega})\} = \phi(\omega) = \alpha\omega + \beta$$
then
$$\arg\{Y(e^{j\omega})\} = \arg\{X(e^{j\omega})\} + \alpha\omega + \beta$$
- The filter bank is said to be phase-preserving but exhibits magnitude distortion

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Perfect Reconstruction Condition

- If an alias-free filter bank has no magnitude and phase distortion, then it is called a perfect reconstruction (PR) QMF bank
- In such a case, we must have
$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2z^{-\ell}$$
- That is, $T(z) = z^{-\ell}$, with ℓ a positive integer, resulting in
$$Y(z) = z^{-\ell}X(z)$$

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Perfect Reconstruction Condition

- In the time-domain, the input-output relation for all possible inputs is given by
$$y[n] = x[n-\ell]$$
indicating that the reconstructed output $y[n]$ is a delayed replica of the input $x[n]$

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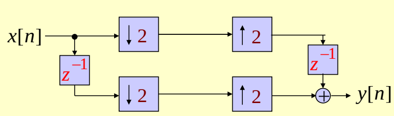
Alias-Free Filter Bank Example

- Thus, for a perfect reconstruction QMF bank, the output is a scaled, delayed replica of the input
- Example - Consider the system shown below



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Alias-Free Filter Bank Example

- Comparing this structure with the general QMF bank structure we conclude that here we have
$$H_0(z) = 1,\ \ H_1(z) = z^{-1},\ \ G_0(z) = z^{-1},\ \ G_1(z) = 1$$
- Substituting these values in the expressions for $T(z)$ and $A(z)$ we get
$$T(z) = \tfrac{1}{2}(z^{-1} + z^{-1}) = z^{-1}$$
$$A(z) = \tfrac{1}{2}(z^{-1} - z^{-1}) = 0$$

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Alias-Free Filter Bank Example

- Thus the simple multirate structure is an alias-free perfect reconstruction filter bank
- However, the filters in the bank do not provide any frequency selectivity

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Product Filters

- The perfect reconstruction condition

$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2z^{-\ell}$$

involves the sum of two product filters:

$$\Pi_0(z) = H_0(z)G_0(z), \quad \Pi_1(z) = H_1(z)G_1(z)$$

- The product filter $\Pi_0(z)$ is a lowpass filter, as both $H_0(z)$ and $G_0(z)$ are lowpass filters
- Likewise, the product filter $\Pi_1(z)$ is a highpass filter, as both $H_1(z)$ and $G_1(z)$ are highpass filters

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A Simple Perfect Reconstruction QMF Bank

- Consider a 2-channel QMF bank with analysis filters given by

$$H_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1}), \quad H_1(z) = \frac{1}{\sqrt{2}}(1-z^{-1})$$

and synthesis filters given by

$$G_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1}), \quad G_1(z) = \frac{1}{\sqrt{2}}(-1+z^{-1})$$

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A Simple Perfect Reconstruction QMF Bank

- **Note**: $H_0(z)$ and $G_0(z)$ are very simple first-order FIR lowpass filters, and $H_1(z)$ and $G_1(z)$ are very simple first-order FIR highpass filters
- Here, $H_0(-z)G_0(z) + H_1(-z)G_1(z)$

$$= \tfrac{1}{2}(1-z^{-1})(1+z^{-1}) + \tfrac{1}{2}(1+z^{-1})(-1+z^{-1})$$
$$= \tfrac{1}{2}(1-z^{-2}) + \tfrac{1}{2}(-1+z^{-2}) = 0$$

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A Simple Perfect Reconstruction QMF Bank

- ➡ The simple 2-channel QMF bank is an alias-free system
- The distortion function is given by

$$T(z) = \tfrac{1}{2}\{H_0(z)H_1(-z) + H_1(z)H_0(-z)\}$$
$$= \tfrac{1}{4}\{(1+z^{-1})(1+z^{-1}) + (1-z^{-1})(-1+z^{-1})\}$$
$$= 2z^{-1}$$

- ➡ The 2-channel QMF bank is a perfect reconstruction system

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A Simple Perfect Reconstruction QMF Bank

- In the time-domain, the outputs of the analysis filters are related to the input $x[n]$ of the QMF bank as

$$u_0[n] = \tfrac{1}{\sqrt{2}}(x[n] + x[n-1])$$
$$u_1[n] = \tfrac{1}{\sqrt{2}}(x[n] - x[n-1])$$

- The scale factor $1/\sqrt{2}$ preserves the energy, i.e.

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A Perfect Reconstruction QMF Bank

$$u_0^2[n]+u_1^2[n]$$
$$=\tfrac{1}{2}\left[(x[n]+x[n-1])^2+(x[n]-x[n-1])^2\right]$$
$$=\tfrac{1}{2}\left(2x^2[n]+2x^2[n-1]\right)$$
$$=x^2[n]+x^2[n-1]$$

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A Simple Perfect Reconstruction QMF Bank

- The input-output relation of the analysis filter bank can be written in matrix form as

$$\begin{bmatrix} u_0[n] \\ u_1[n] \end{bmatrix} = \mathbf{H}_2 \begin{bmatrix} x[n] \\ x[n-1] \end{bmatrix}$$

where

$$\mathbf{H}_2 = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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A Simple Perfect Reconstruction QMF Bank

$\mathbf{H}_2$ is seen to be the 2×2 Haar transform matrix except for the scale factor $1/\sqrt{2}$

- **Note**: $\mathbf{H}_2$ is a symmetric orthonormal matrix, and hence

$$\mathbf{H}_2^{-1} = \mathbf{H}_2^t = \mathbf{H}_2$$

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A Simple Perfect Reconstruction QMF Bank

- The reconstruction of the input data $x[n]$ and $x[n-1]$ from the output data $u_0[n]$ and $u_1[n]$ is then given by

$$\begin{bmatrix} x[n] \\ x[n-1] \end{bmatrix} = \mathbf{H}_2^t \begin{bmatrix} u_0[n] \\ u_1[n] \end{bmatrix}$$

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An Alias-Free Realization

- The aliasing cancellation condition

$$H_0(-z)G_0(z)+H_1(-z)G_1(z)=0$$

is satisfied if we choose

$$G_0(z)=H_1(-z), \quad G_1(z)=-H_0(-z)$$

- In this case, we have

$$\Pi_1(z)=-H_0(-z)H_1(z)=-\Pi_0(-z)$$

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An Alias-Free Realization

- The perfect reconstruction condition can then be written as

$$\Pi_0(z)-\Pi_0(-z)=2z^{-\ell}$$

- Since the even powers of z in $\Pi_0(z)$ are cancelled by the even powers of z in $\Pi_0(-z)$, ℓ must be an odd integer

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An Alias-Free Realization

- Moreover, $\Pi_0(z)$ cannot contain any terms with odd powers of z except the term involving $z^{-\ell}$ whose coefficient is simply 1
- Under the alias-free condition

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$$

the algorithm for the design of a perfect reconstruction 2-channel QMF bank thus involves 3 steps:

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An Alias-Free Realization

- (1) The design of a lowpass product filter $\Pi_0(z)$ satisfying the condition

$$\Pi_0(z) - \Pi_0(-z) = 2z^{-\ell}$$

- (2) Factorization of $\Pi_0(z)$ to determine the lowpass filters $H_0(z)$ and $G_0(z)$
- (3) Determination of the highpass filters $H_1(z)$ and $G_1(z)$ by using

$$G_0(z) = H_1(-z), \quad G_1(z) = -H_0(-z)$$

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An Alias-Free Realization

- A more convenient form of the perfect reconstruction condition is obtained by multiplying both sides of

$$\Pi_0(z) - \Pi_0(-z) = 2z^{-\ell}$$

with z^{ℓ}, resulting in

$$z^{\ell}\Pi_0(z) - z^{\ell}\Pi_0(-z) = 2$$

- Define a normalized form of the product filter $\Pi(z) = z^{\ell}\Pi_0(z)$

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An Alias-Free Realization

- Now, $\Pi(-z) = (-z)^{\ell}\Pi_0(-z) = -z^{\ell}\Pi_0(-z)$, as ℓ is an odd integer
- ➡ The perfect reconstruction condition can be rewritten as $\Pi(z) + \Pi(-z) = 2$
- Thus, $\Pi(z)$ is a zero-phase half-band lowpass filter whose constant term, that is, the coefficient of z^0 is 1

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An Alias-Free Realization

- A very simple factorization of the product filter $\Pi_0(z)$ is obtained by choosing

$$G_0(z) = H_0(z)$$

- Then, $H_1(z) = G_0(-z) = H_0(-z)$

$$G_1(z) = -H_1(z) = -H_0(-z)$$

- In the case of a real coefficient filter, we thus have

$$\left|H_1(e^{j\omega})\right| = \left|H_0(e^{j(\pi-\omega)})\right|$$

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An Alias-Free Realization

- ➡ If $H_0(z)$ is a lowpass filter, then $H_1(z)$ is a highpass filter, and vice versa
- In fact, $\left|H_1(e^{j\omega})\right| = \left|H_0(e^{j(\pi-\omega)})\right|$ indicates that $\left|H_1(e^{j\omega})\right|$ is a mirror image of $\left|H_0(e^{j\omega})\right|$ with respect to $\pi/2$, the quadrature frequency
- This has given rise to the name quadrature-mirror filter bank

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An Alias-Free Realization

- The distortion transfer function $T(z)$ in this case reduces to

$$T(z)=\tfrac{1}{2}\{H_0^2(z)-H_1^2(z)\}=\tfrac{1}{2}\{H_0^2(z)-H_0^2(-z)\}$$

- A computationally efficient realization of the above alias-free 2-channel QMF bank is obtained by realizing the analysis and synthesis filters in polyphase form

An Alias-Free Realization

- Let the 2-band Type 1 polyphase representation of $H_0(z)$ be given by

$$H_0(z)=E_0(z^2)+z^{-1}E_1(z^2)$$

- Then,

$$H_1(z)=H_0(-z)=E_0(z^2)-z^{-1}E_1(z^2)$$

- In matrix form we have

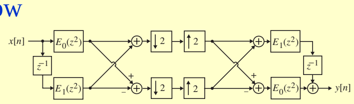
$$\begin{bmatrix}H_0(z)\\ H_1(z)\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}E_0(z^2)\\ z^{-1}E_1(z^2)\end{bmatrix}$$

An Alias-Free Realization

- Likewise, the synthesis filter can be expressed in matrix form as

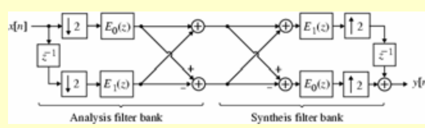
$$[G_0(z)\quad G_1(z)]=\begin{bmatrix}z^{-1}E_1(z^2) & E_0(z^2)\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

- The polyphase realization of the alias-free 2-channel QMF bank is thus as shown below



An Alias-Free Realization

- Making use of the cascade equivalences, the above structure can be further simplified as shown below



An Alias-Free Realization

- Substituting the polyphase representations of the analysis filters we arrive at the expression for the distortion function $T(z)$ in terms of the polyphase components as

$$T(z)=2z^{-1}E_0(z^2)E_1(z^2)$$

Alias-Free FIR QMF Bank

- If in the above alias-free QMF bank $H_0(z)$ is a linear-phase FIR filter, then its polyphase components $E_0(z)$ and $E_1(z)$, are also linear-phase FIR transfer functions
- In this case, $T(z)=2z^{-1}E_0(z^2)E_1(z^2)$ exhibits a linear-phase characteristics
- As a result, the corresponding 2-channel QMF bank has no phase distortion

Alias-Free FIR QMF Bank

- However, in general $|T(e^{j\omega})|$ is not a constant, and as a result, the QMF bank exhibits magnitude distortion
- We next outline a method to minimize the residual amplitude distortion
- Let $H_0(z)$ be a length-N real-coefficient linear-phase FIR transfer function:

$$H_0(z) = \sum_{n=0}^{N-1} h_0[n] z^{-n}$$

Alias-Free FIR QMF Bank

- Note: $H_0(z)$ can either be a Type 1 or a Type 2 linear-phase FIR transfer function since it has to be a lowpass filter
- Then $h_0[n]$ satisfy the condition

$$h_0[n] = h_0[N-n]$$

- In this case we can write

$$H_0(e^{j\omega}) = e^{j\omega N/2} \tilde{H}_0(\omega)$$

- In the above $\tilde{H}_0(\omega)$ is the amplitude function, a real function of ω

Alias-Free FIR QMF Bank

- The frequency response of the distortion transfer function can now be written as

$$T(e^{j\omega}) = \frac{e^{-jN\omega}}{2} \{ |H_0(e^{j\omega})|^2 - (-1)^N |H_0(e^{j(\pi-\omega)})|^2 \}$$

- From the above, it can be seen that if N is even, then $T(e^{j\omega}) = 0$ at $\omega = \pi/2$, implying severe amplitude distortion at the output of the filter bank

Alias-Free FIR QMF Bank

- $\Rightarrow$ N must be odd, in which case we have

$$T(e^{j\omega}) = \frac{e^{-jN\omega}}{2} \{ |H_0(e^{j\omega})|^2 + |H_0(e^{j(\pi-\omega)})|^2 \}$$

$$= \frac{e^{-jN\omega}}{2} \{ |H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 \}$$

- It follows from the above that the FIR 2-channel QMF bank will be of perfect reconstruction type if

$$|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$$

Alias-Free FIR QMF Bank

- Now, the 2-channel QMF bank with linear-phase filters has no phase distortion, but will always exhibit amplitude distortion unless $|T(e^{j\omega})|$ is a constant for all ω
- If $H_0(z)$ is a very good lowpass filter with $|H_0(e^{j\omega})| \cong 1$ in the passband and $|H_0(e^{j\omega})| \cong 0$ in the stopband, then $H_1(z)$ is a very good highpass filter with its passband coinciding with the stopband of $H_0(z)$, and vice-versa

Alias-Free FIR QMF Bank

- As a result, $|T(e^{j\omega})| \cong 1/2$ in the passbands of $H_0(z)$ and $H_1(z)$
- $\Rightarrow$ Amplitude distortion occurs primarily in the transition band of these filters
- Degree of distortion determined by the amount of overlap between their squared-magnitude responses

Alias-Free FIR QMF Bank

- This distortion can be minimized by controlling the overlap, which in turn can be controlled by appropriately choosing the passband edge of $H_0(z)$
- One way to minimize the amplitude distortion is to iteratively adjust the filter coefficients $h_0[n]$ of $H_0(z)$ on a computer such that

$$|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 \cong 1$$

is satisfied for all values of ω

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Alias-Free FIR QMF Bank

- To this end, the objective function ϕ to be minimized can be chosen as a linear combination of two functions:

(1) stopband attenuation of $H_0(z)$, and

(2) sum of squared magnitude responses of $H_0(z)$ and $H_1(z)$

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Alias-Free FIR QMF Bank

- One such objective function is given by

$$\Phi = \alpha\Phi_1 + (1-\alpha)\Phi_2$$

where

$$\Phi_1 = \int_{\omega_s}^{\pi} \left|H(e^{j\omega})\right|^2 d\omega$$

and

$$\Phi_2 = \int_0^{\pi} \left(1 - \left|H_0(e^{j\omega})\right|^2 - \left|H_1(e^{j\omega})\right|^2\right)^2 d\omega$$

and $0 < \alpha < 1$, and $\omega_s = \frac{\pi}{2} + \varepsilon$ for some small $\varepsilon > 0$

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Alias-Free FIR QMF Bank

- Since $|T(e^{j\omega})|$ is symmetric with respect to $\pi/2$, the second integral in the objective function Φ can be replaced with

$$\Phi_2 = 2\int_0^{\pi/2} \left(1 - \left|H_0(e^{j\omega})\right|^2 - \left|H_1(e^{j\omega})\right|^2\right)^2 d\omega$$

- After Φ has been made very small by the optimization procedure, both Φ_1 and Φ_2 will also be very small

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Alias-Free FIR QMF Bank

- Using this approach, Johnston has designed a large class of linear-phase FIR filters meeting a variety of specifications and has tabulated their impulse response coefficients
- Program 14_1 can be used to verify the performance of Johnston's filters

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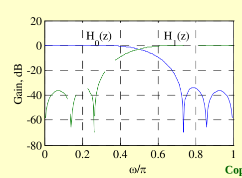
Alias-Free FIR QMF Bank

- Example - The gain responses of the length-12 linear-phase FIR lowpass filter 12B and its power-complementary highpass filter obtained using Program 14_1 are shown below



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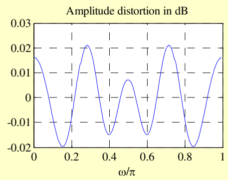
Alias-Free FIR QMF Bank

- The program then computes the amplitude distortion $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2$ in dB as shown below



Alias-Free FIR QMF Bank

- From the gain response plot it can be seen that the stopband edge ω_s of the lowpass filter 12B is about 0.71π, which corresponds to a transition bandwidth of

$$(\omega_s - 0.5\pi)/2 = 0.105\pi$$

- The minimum stopband attenuation is approximately 34 dB

Alias-Free FIR QMF Bank

- The amplitude distortion function is very close to 0 dB in both the passbands and the stopbands of the two filters, with a peak value of ± 0.02 dB

Alias-Free IIR QMF Bank

- Let the polyphase components $E_0(z)$ and $E_1(z)$ of $H_0(z)$ be expressed as

$$E_0(z) = \tfrac{1}{2}\mathcal{A}_0(z), \quad E_1(z) = \tfrac{1}{2}\mathcal{A}_1(z)$$

with $\mathcal{A}_0(z)$ and $\mathcal{A}_1(z)$ being stable allpass functions

- Thus, $H_0(z) = \tfrac{1}{2}\left[\mathcal{A}_0(z^2) + z^{-1}\mathcal{A}_1(z^2)\right]$

$$H_1(z) = \tfrac{1}{2}\left[\mathcal{A}_0(z^2) - z^{-1}\mathcal{A}_1(z^2)\right]$$

Alias-Free IIR QMF Bank

- In matrix form, the analysis filters can be expressed as

$$\begin{bmatrix} H_0(z) \\ H_0(z) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \mathcal{A}_0(z^2) \\ z^{-1}\mathcal{A}_1(z^2) \end{bmatrix}$$

- The corresponding synthesis filters in matrix form are given by

$$[G_0(z) \quad G_1(z)] = \frac{1}{2}\left[z^{-1}\mathcal{A}_1(z^2) \quad \mathcal{A}_0(z^2)\right]\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

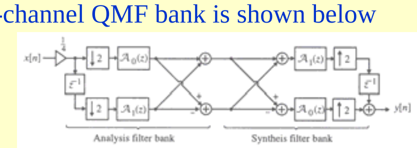
Alias-Free IIR QMF Bank

- Thus, the synthesis filters are given by

$$G_0(z) = \tfrac{1}{2}[\mathcal{A}_0(z^2) + z^{-1}\mathcal{A}_1(z^2)] = H_0(z)$$

$$G_1(z) = \tfrac{1}{2}[-\mathcal{A}_0(z^2) + z^{-1}\mathcal{A}_1(z^2)] = -H_1(z)$$

- The realization of the magnitude-preserving 2-channel QMF bank is shown below



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- From
  $$H_0(z) = \tfrac{1}{2}\left[\mathcal{A}_0(z^2) + z^{-1}\mathcal{A}_1(z^2)\right]$$
  it can be seen that the lowpass transfer function $H_0(z)$ has a polyphase-like decomposition, except here the polyphase components are stable allpass transfer functions

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- It has been shown earlier that a bounded-real (BR) transfer function $H_0(z) = P_0(z)/D(z)$ of odd order, with no common factors between its numerator and denominator, can be expressed in the form
  $$H_0(z) = \tfrac{1}{2}\left[\mathcal{A}_0(z^2) + z^{-1}\mathcal{A}_1(z^2)\right]$$
  if it satisfies the power-symmetry condition
  $$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 1$$
  and has a symmetric numerator $P_0(z)$

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- It has also been shown that any odd-order elliptic lowpass half-band filter $H_0(z)$ with a frequency response specification given by
  $$1-\delta_p \le |H(e^{j\omega})| \le 1, \text{ for } 0 \le \omega \le \omega_p$$
  $$|H(e^{j\omega})| \le \delta_s, \text{ for } \omega_s \le \omega \le \pi$$
  and satisfying the conditions $\omega_p + \omega_s = \pi$ and $\delta_s^2 = 4\delta_p(1-\delta_p)$ can always be expressed in the form
  $$H_0(z) = \tfrac{1}{2}\left[\mathcal{A}_0(z^2) + z^{-1}\mathcal{A}_1(z^2)\right]$$

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- The poles of the elliptic filter satisfying the two conditions on bandedges and ripples lie on the imaginary axis
- Using the pole-interlacing property discussed earlier, on can readily identify the expressions for the two allpass transfer functions $\mathcal{A}_0(z)$ and $\mathcal{A}_1(z)$

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- Example - The frequency response specifications of a real-coefficient lowpass half-band filter are given by: $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, and $\delta_s = 0.0155$
- From $\delta_s^2 = 4\delta_p(1-\delta_p)$ we get $\delta_p = 0.00012013$
- In dB, the passband and stopband ripples are `Rp = 0.0010435178` and `Rs = 36.193366`

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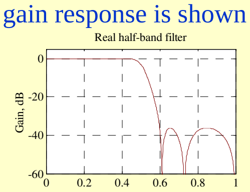
- Using the M-file `ellipord` we determine the minimum order of the elliptic lowpass filter to be 5
- Next, using the M-file `ellip` the transfer function of the lowpass filter is determined whose gain response is shown below



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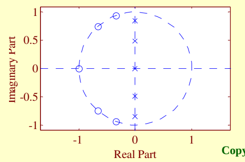
- The poles obtained using the function `tf2zp` are at $z = 0$, $z = \pm\, j0.486625263$, and $z = \pm\, j0.486625263$
- The pole-zero plot obtained using `zplane` is shown below



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- Using the pole-interlacing property we arrive at the transfer functions of the two allpass filters as given below:

$$\mathcal{A}_0(z^2) = \frac{z^{-2} + 0.2368041466}{1 + 0.2368041466 z^{-2}}$$

$$\mathcal{A}_1(z^2) = \frac{z^{-2} + 0.7149039978}{1 + 0.7149039978 z^{-2}}$$

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