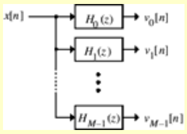
Digital Filter Banks

- The digital filter bank is set of bandpass filters with either a common input or a summed output
- An M-band analysis filter bank is shown below



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Digital Filter Banks

- The subfilters $H_k(z)$ in the analysis filter bank are known as analysis filters
- The analysis filter bank is used to decompose the input signal $x[n]$ into a set of subband signals $v_k[n]$ with each subband signal occupying a portion of the original frequency band

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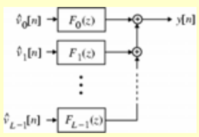
Digital Filter Banks

- An L-band synthesis filter bank is shown below



- It performs the dual operation to that of the analysis filter bank

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Digital Filter Banks

- The subfilters $F_k(z)$ in the synthesis filter bank are known as synthesis filters
- The synthesis filter bank is used to combine a set of subband signals $\hat{v}_k[n]$ (typically belonging to contiguous frequency bands) into one signal $y[n]$ at its output

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Uniform Digital Filter Banks

- A simple technique to design a class of filter banks with equal passband widths is outlined next
- Let $H_0(z)$ represent a causal lowpass digital filter with a real impulse response $h_0[n]$:
$$H_0(z) = \sum_{n=-\infty}^{\infty} h_0[n] z^{-n}$$
- The filter $H_0(z)$ is assumed to be an IIR filter without any loss of generality

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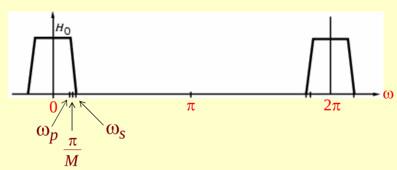
Uniform Digital Filter Banks

- Assume that $H_0(z)$ has its passband edge ω_p and stopband edge ω_s around π/M, where M is some arbitrary integer, as indicated below



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Uniform Digital Filter Banks

- Now, consider the transfer function $H_k(z)$ whose impulse response $h_k[n]$ is given by
$$h_k[n] = h_0[n]e^{j2\pi kn/M} = h_0[n]W_M^{-kn},$$
$$0 \le k \le M-1$$
where we have used the notation $W_M = e^{-j2\pi/M}$
- Thus,
$$H_k(z) = \sum_{n=-\infty}^{\infty} h_k[n]z^{-n} = \sum_{n=-\infty}^{\infty} h_0[n]\left(zW_M^k\right)^{-n},$$
$$0 \le k \le M-1$$

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Uniform Digital Filter Banks

- i.e.,
$$H_k(z) = H_0(zW_M^k), \quad 0 \le k \le M-1$$
- The corresponding frequency response is given by
$$H_k(e^{j\omega}) = H_0(e^{j(\omega - 2\pi k/M)}), \quad 0 \le k \le M-1$$
- Thus, the frequency response of $H_k(z)$ is obtained by shifting the response of $H_0(z)$ to the right by an amount $2\pi k/M$

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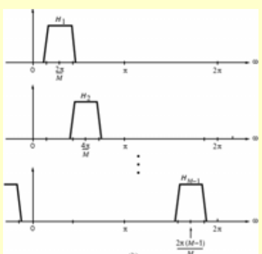
Uniform Digital Filter Banks

- The responses of $H_k(z)$, $H_k(z)$, . . . , $H_k(z)$ are shown below



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Uniform Digital Filter Banks

- Note: The impulse responses $h_k[n]$ are, in general complex, and hence $|H_k(e^{j\omega})|$ does not necessarily exhibit symmetry with respect to $\omega = 0$
- The responses shown in the figure of the previous slide can be seen to be uniformly shifted version of the response of the basic prototype filter $H_0(z)$

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Uniform Digital Filter Banks

- The M filters defined by
$$H_k(z) = H_0(zW_M^k), \quad 0 \le k \le M-1$$
could be used as the analysis filters in the analysis filter bank or as the synthesis filters in the synthesis filter bank
- Since the magnitude responses of all M filters are uniformly shifted version of that of the prototype filter, the filter bank obtained is called a uniform filter bank

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Uniform DFT Filter Banks

Polyphase Implementation

- Let the prototype lowpass transfer function be represented in its M-band polyphase form:
$$H_0(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_\ell(z^M)$$
where $E_\ell(z)$ is the ℓ-th polyphase component of $H_0(z)$:
$$E_\ell(z) = \sum_{n=0}^{\infty} e_\ell[n]z^{-n} = \sum_{n=0}^{\infty} h_0[\ell + nM]z^{-n},$$
$$0 \le \ell \le M-1$$

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Uniform DFT Filter Banks

- Substituting z with zW_M^k in the expression for $H_0(z)$ we arrive at the M-band polyphase decomposition of $H_k(z)$:

$$H_k(z) = \sum_{\ell=0}^{M-1} z^{-\ell} W_M^{-k\ell} E_\ell(z^M W_M^{kM})$$
$$= \sum_{\ell=0}^{M-1} z^{-\ell} W_M^{-k\ell} E_\ell(z^M), \quad 0 \le k \le M-1$$

- In deriving the last expression we have used the identity $W_M^{kM} = 1$

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Uniform DFT Filter Banks

- The equation on the previous slide can be written in matrix form as

$$H_k(z) = [1 \;\; W_M^{-k} \;\; W_M^{-2k} \;\; \cdots \;\; W_M^{-(M-1)k}] \begin{bmatrix} E_0(z^M) \\ z^{-1}E_1(z^M) \\ z^{-2}E_2(z^M) \\ \vdots \\ z^{-(M-1)}E_{M-1}(z^M) \end{bmatrix}$$

$$0 \le k \le M-1$$

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Uniform DFT Filter Banks

- All M equations on the previous slide can be combined into one matrix equation as

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_M^{-1} & W_M^{-2} & \cdots & W_M^{-(M-1)} \\ 1 & W_M^{-2} & W_M^{-4} & \cdots & W_M^{-2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_M^{-(M-1)} & W_M^{-2(M-1)} & \cdots & W_M^{-(M-1)^2} \end{bmatrix}}_{M\,\mathbf{D}^{-1}} \begin{bmatrix} E_0(z^M) \\ z^{-1}E_1(z^M) \\ z^{-2}E_2(z^M) \\ \vdots \\ z^{-(M-1)}E_{M-1}(z^M) \end{bmatrix}$$

- In the above $\mathbf{D}$ is the $M \times M$ DFT matrix

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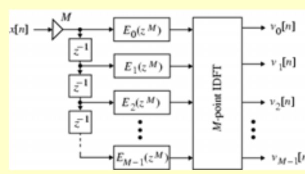
Uniform DFT Filter Banks

- An efficient implementation of the M-band uniform analysis filter bank, more commonly known as the uniform DFT analysis filter bank, is then as shown below



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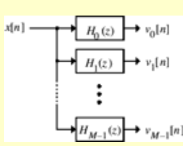
Uniform DFT Filter Banks

- The computational complexity of an M-band uniform DFT filter bank is much smaller than that of a direct implementation as shown below



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Uniform DFT Filter Banks

- For example, an M-band uniform DFT analysis filter bank based on an N-tap prototype lowpass filter requires a total of $\frac{M}{2}\log_2 M + N$ multipliers
- On the other hand, a direct implementation requires NM multipliers

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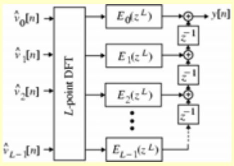
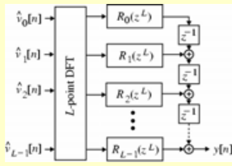
Uniform DFT Filter Banks

- Following a similar development, we can derive the structure for a uniform DFT synthesis filter bank as shown below



Type I uniform DFT synthesis filter bank



Type II uniform DFT synthesis filter bank

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Uniform DFT Filter Banks

- **Example** – Using Program 13_8 we design a length-23 4th-band linear-phase FIR filter whose polyphase components are given by

$$E_0(z) = 0.0016369 - 0.01121888z^{-1} + 0.06311487z^{-2} + 0.220885z^{-3} - 0.0272555z^{-4} + 0.0038269z^{-5}$$

$$E_1(z) = 0.0031396 - 0.0251749z^{-1} + 0.147533z^{-2} + 0.1475329z^{-3} - 0.0251749z^{-4} + 0.0031396z^{-5}$$

$$E_2(z) = 0.003827 - 0.0272555z^{-1} + 0.220885z^{-2} + 0.631149z^{-3} - 0.0112189z^{-4} + 0.001637z^{-5}$$

$$E_3(z) = 0.25z^{-2}$$

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Uniform DFT Filter Banks

- The 4 filters of the 4-channel uniform DFT filter bank are then given by

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ H_3(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} E_0(z^4) \\ z^{-1}E_1(z^4) \\ z^{-2}E_2(z^4) \\ z^{-3}E_3(z^4) \end{bmatrix}$$

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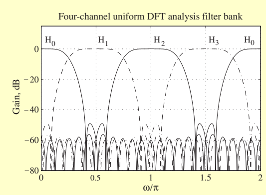
Uniform DFT Filter Banks

- A plot of the gain responses of the 4 analysis filters is shown below:



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Uniform DFT Filter Banks

- Now $E_i(z^M)$ can be expressed in terms of

$$\begin{bmatrix} E_0(z^M) \\ z^{-1}E_1(z^M) \\ z^{-2}E_2(z^M) \\ \vdots \\ z^{-(M-1)}E_{M-1}(z^M) \end{bmatrix} = \frac{1}{M}\mathbf{D} \begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix}$$

- The above equation can be used to determine the polyphase components of an IIR transfer function $H_0(z)$

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Uniform DFT Filter Banks

- **Example** – We develop a 3-band polyphase decomposition of the transfer function

$$H(z) = \frac{a + bz^{-1}}{1 + cz^{-1}}, \qquad |c| < 1$$

- From the matrix equation given in the previous slide we have

$$\begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^1 & W_3^2 \\ 1 & W_3^2 & W_3^1 \end{bmatrix} \begin{bmatrix} H(z) \\ H(zW_3^1) \\ H(zW_3^2) \end{bmatrix}$$

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Uniform DFT Filter Banks

- Therefore,

$$E_0(z^3) = \tfrac{1}{3}[H(z) + H(zW_3^1) + H(zW_3^2)]$$
$$= \frac{a + bc^2 z^{-3}}{1 + c^3 z^{-3}}$$
$$z^{-1}E_1(z^3) = \tfrac{1}{3}[H(z) + W_3^1 H(zW_3^1) + W_3^2 H(zW_3^2)]$$
$$= z^{-1}\left(\frac{b - ac}{1 + c^3 z^{-3}}\right)$$
$$z^{-2}E_2(z^3) = \tfrac{1}{3}[H(z) + W_3^2 H(zW_3^1) + W_3^1 H(zW_3^2)]$$
$$= z^{-2}\left(\frac{-bc + ac^2}{1 + c^3 z^{-3}}\right)$$

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