

## DTFT Properties

- **Example** - Determine the DTFT  $Y(e^{j\omega})$  of

$$y[n] = (n+1)\alpha^n \mu[n], \quad |\alpha| < 1$$

- Let  $x[n] = \alpha^n \mu[n], |\alpha| < 1$

- We can therefore write

$$y[n] = nx[n] + x[n]$$

- From Table 3.3, the DTFT of  $x[n]$  is given by

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

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## DTFT Properties

- Using the differentiation property of the DTFT given in Table 3.2, we observe that the DTFT of  $nx[n]$  is given by

$$j \frac{dX(e^{j\omega})}{d\omega} = j \frac{d}{d\omega} \left( \frac{1}{1 - \alpha e^{-j\omega}} \right) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$$

- Next using the linearity property of the DTFT given in Table 3.4 we arrive at

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2} + \frac{1}{1 - \alpha e^{-j\omega}} = \frac{1}{(1 - \alpha e^{-j\omega})^2}$$

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## DTFT Properties

- **Example** - Determine the DTFT  $V(e^{j\omega})$  of the sequence  $v[n]$  defined by

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

- From Table 3.3, the DTFT of  $\delta[n]$  is 1
- Using the time-shifting property of the DTFT given in Table 3.4 we observe that the DTFT of  $\delta[n-1]$  is  $e^{-j\omega}$  and the DTFT of  $v[n-1]$  is  $e^{-j\omega}V(e^{j\omega})$

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## DTFT Properties

- Using the linearity property of Table 3.4 we then obtain the frequency-domain representation of

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

as

$$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$$

- Solving the above equation we get

$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}$$

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## Energy Density Spectrum

- The total energy of a finite-energy sequence  $g[n]$  is given by

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} |g[n]|^2$$

- From Parseval's relation given in Table 3.4 we observe that

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} |g[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{j\omega})|^2 d\omega$$

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## Energy Density Spectrum

- The quantity

$$S_{gg}(\omega) = |G(e^{j\omega})|^2$$

is called the **energy density spectrum**

- The area under this curve in the range  $-\pi \leq \omega \leq \pi$  divided by  $2\pi$  is the energy of the sequence

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## Band-limited Discrete-time Signals

- Since the spectrum of a discrete-time signal is a periodic function of  $\omega$  with a period  $2\pi$ , a full-band signal has a spectrum occupying the frequency range  $-\pi \leq \omega \leq \pi$
- A band-limited discrete-time signal has a spectrum that is limited to a portion of the frequency range  $-\pi \leq \omega \leq \pi$

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## Band-limited Discrete-time Signals

- An ideal band-limited signal has a spectrum that is zero outside a frequency range  $0 < \omega_a \leq |\omega| \leq \omega_b < \pi$ , that is

$$X(e^{j\omega}) = \begin{cases} 0, & 0 \leq |\omega| < \omega_a \\ 0, & \omega_b < |\omega| < \pi \end{cases}$$

- An ideal band-limited discrete-time signal cannot be generated in practice

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## Band-limited Discrete-time Signals

- A classification of a band-limited discrete-time signal is based on the frequency range where most of the signal's energy is concentrated
- A lowpass discrete-time real signal has a spectrum occupying the frequency range  $0 < |\omega| \leq \omega_p < \pi$  and has a bandwidth of  $\omega_p$

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## Band-limited Discrete-time Signals

- A highpass discrete-time real signal has a spectrum occupying the frequency range  $0 < \omega_p \leq |\omega| < \pi$  and has a bandwidth of  $\pi - \omega_p$
- A bandpass discrete-time real signal has a spectrum occupying the frequency range  $0 < \omega_L \leq |\omega| \leq \omega_H < \pi$  and has a bandwidth of  $\omega_H - \omega_L$

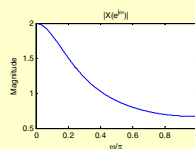
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## Band-limited Discrete-time Signals

- Example – Consider the sequence  $x[n] = (0.5)^n \mu[n]$
- Its DTFT is given below on the left along with its magnitude spectrum shown below on the right

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$



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## Band-limited Discrete-time Signals

- It can be shown that 80% of the energy of this lowpass signal is contained in the frequency range  $0 \leq |\omega| \leq 0.5081\pi$
- Hence, we can define the 80% bandwidth to be  $0.5081\pi$  radians

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## Energy Density Spectrum

- **Example** - Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- Here

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_{LP}(e^{j\omega})|^2 d\omega$$

where

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

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## Energy Density Spectrum

- Therefore

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

- Hence,  $h_{LP}[n]$  is a finite-energy lowpass sequence

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## DTFT Computation Using MATLAB

- The function `freqz` can be used to compute the values of the DTFT of a sequence, described as a rational function in the form of

$$X(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega} + \dots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \dots + d_N e^{-j\omega N}}$$

at a prescribed set of discrete frequency points  $\omega = \omega_\ell$

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## DTFT Computation Using MATLAB

- For example, the statement  
`H = freqz(num,den,w)`  
returns the frequency response values as a vector `H` of a DTFT defined in terms of the vectors `num` and `den` containing the coefficients  $\{p_i\}$  and  $\{d_i\}$ , respectively at a prescribed set of frequencies between 0 and  $2\pi$  given by the vector `w`

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## DTFT Computation Using MATLAB

- There are several other forms of the function `freqz`
- Program 3\_1.m in the text can be used to compute the values of the DTFT of a real sequence
- It computes the real and imaginary parts, and the magnitude and phase of the DTFT

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## DTFT Computation Using MATLAB

- **Example** - Plots of the real and imaginary parts, and the magnitude and phase of the DTFT

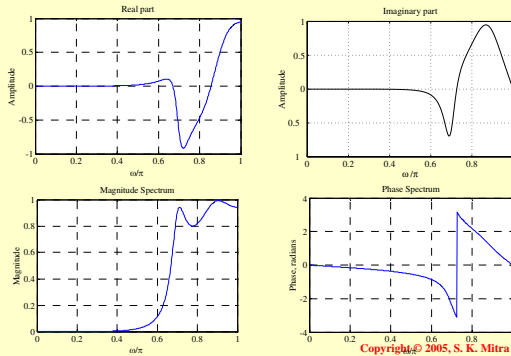
$$X(e^{j\omega}) = \frac{0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} - 0.033e^{-j3\omega} + 0.008e^{-j4\omega}}{1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} + 1.6e^{-j3\omega} + 0.41e^{-j4\omega}}$$

are shown on the next slide

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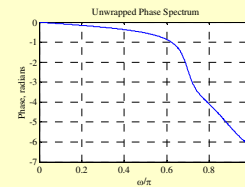
## DTFT Computation Using MATLAB



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## DTFT Computation Using MATLAB

- **Note:** The phase spectrum displays a discontinuity of  $2\pi$  at  $\omega = 0.72$
- This discontinuity can be removed using the function `unwrap` as indicated below



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## Linear Convolution Using DTFT

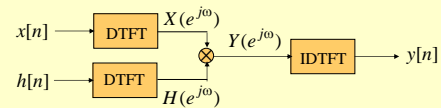
- An important property of the DTFT is given by the convolution theorem in Table 3.4
- It states that if  $y[n] = x[n] \otimes h[n]$ , then the DTFT  $Y(e^{j\omega})$  of  $y[n]$  is given by
 
$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$
- An implication of this result is that the linear convolution  $y[n]$  of the sequences  $x[n]$  and  $h[n]$  can be performed as follows:

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## Linear Convolution Using DTFT

- 1) Compute the DTFTs  $X(e^{j\omega})$  and  $H(e^{j\omega})$  of the sequences  $x[n]$  and  $h[n]$ , respectively
- 2) Form the DTFT  $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
- 3) Compute the IDFT  $y[n]$  of  $Y(e^{j\omega})$



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## The Unwrapped Phase Function

- In numerical computation, when the computed phase function is outside the range  $[-\pi, \pi]$ , the phase is computed modulo  $2\pi$ , to bring the computed value to this range
- Thus, the phase functions of some sequences exhibit discontinuities of  $2\pi$  radians in the plot

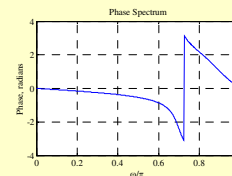
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## The Unwrapped Phase Function

- For example, there is a discontinuity of  $2\pi$  at  $\omega = 0.72$  in the phase response below

$$X(e^{j\omega}) = \frac{0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} - 0.033e^{-j3\omega} + 0.008e^{-j4\omega}}{1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} + 1.6e^{-j3\omega} + 0.41e^{-j4\omega}}$$



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## The Unwrapped Phase Function

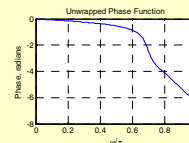
- In such cases, often an alternate type of phase function that is continuous function of  $\omega$  is derived from the original phase function by removing the discontinuities of  $2\pi$
- Process of discontinuity removal is called **unwrapping the phase**
- The unwrapped phase function will be denoted as  $\theta_c(\omega)$

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## The Unwrapped Phase Function

- In MATLAB, the unwrapping can be implemented using the M-file **unwrap**
- The unwrapped phase function of the DTFT of previous page is shown below



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## The Unwrapped Phase Function

- The conditions under which the phase function will be a continuous function of  $\omega$  is next derived

- Now

$$\ln X(e^{j\omega}) = |X(e^{j\omega})| + j\theta(\omega)$$

where

$$\theta(\omega) = \arg\{H(e^{j\omega})\}$$

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## The Unwrapped Phase Function

- If  $\ln X(e^{j\omega})$  exists, then its derivative with respect to  $\omega$  also exists and is given by

$$\begin{aligned} \frac{d \ln X(e^{j\omega})}{d\omega} &= \frac{1}{X(e^{j\omega})} \left[ \frac{dX(e^{j\omega})}{d\omega} \right] \\ &= \frac{1}{X(e^{j\omega})} \left[ \frac{dX_{\text{re}}(e^{j\omega})}{d\omega} + j \frac{dX_{\text{im}}(e^{j\omega})}{d\omega} \right] \end{aligned}$$

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## The Unwrapped Phase Function

- From  $\ln X(e^{j\omega}) = |X(e^{j\omega})| + j\theta(\omega)$ ,  $d \ln X(e^{j\omega}) / d\omega$  is also given by

$$\frac{d \ln X(e^{j\omega})}{d\omega} = \frac{d|X(e^{j\omega})|}{d\omega} + j \frac{d\theta(\omega)}{d\omega}$$

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## The Unwrapped Phase Function

- Thus,  $d\theta(\omega) / d\omega$  is given by the imaginary part of

$$\frac{1}{X(e^{j\omega})} \left[ \frac{dX_{\text{re}}(e^{j\omega})}{d\omega} + j \frac{dX_{\text{im}}(e^{j\omega})}{d\omega} \right]$$

- Hence,

$$\frac{d\theta(\omega)}{d\omega} = \frac{1}{|X(e^{j\omega})|^2} \left[ X_{\text{re}}(e^{j\omega}) \frac{dX_{\text{im}}(e^{j\omega})}{d\omega} - X_{\text{im}}(e^{j\omega}) \frac{dX_{\text{re}}(e^{j\omega})}{d\omega} \right]$$

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## The Unwrapped Phase Function

- The phase function can thus be defined unequivocally by its derivative  $d\theta(\omega)/d\omega$ :

$$\theta(\omega) = \int_0^{\omega} \left[ \frac{d\theta(\eta)}{d\eta} \right] d\eta,$$

with the constraint

$$\theta(0) = 0$$

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## The Unwrapped Phase Function

- The phase function defined by

$$\theta(\omega) = \int_0^{\omega} \left[ \frac{d\theta(\eta)}{d\eta} \right] d\eta$$

is called the **unwrapped phase function** of  $X(e^{j\omega})$  and it is a continuous function of  $\omega$

- $\Rightarrow \ln X(e^{j\omega})$  **exists**

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## The Unwrapped Phase Function

- Moreover, the phase function will be an odd function of  $\omega$  if

$$\frac{1}{\pi} \int_0^{2\pi} \left[ \frac{d\theta(\eta)}{d\eta} \right] d\eta = 0$$

- If the above constraint is not satisfied, then the computed phase function will exhibit absolute jumps greater than  $\pi$

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## The Frequency Response

- Most discrete-time signals encountered in practice can be represented as a linear combination of a very large, maybe infinite, number of sinusoidal discrete-time signals of different angular frequencies
- Thus, knowing the response of the LTI system to a single sinusoidal signal, we can determine its response to more complicated signals by making use of the superposition property

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## The Frequency Response

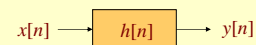
- An important property of an LTI system is that for certain types of input signals, called **eigen functions**, the output signal is the input signal multiplied by a complex constant
- We consider here one such eigen function as the input

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## The Frequency Response

- Consider the LTI discrete-time system with an impulse response  $\{h[n]\}$  shown below



- Its input-output relationship in the time-domain is given by the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

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## The Frequency Response

- If the input is of the form

$$x[n] = e^{j\omega n}, \quad -\infty < n < \infty$$

then it follows that the output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} = \left( \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

- Let

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

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## The Frequency Response

- Then we can write

$$y[n] = H(e^{j\omega}) e^{j\omega n}$$

- Thus for a complex exponential input signal  $e^{j\omega n}$ , the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant  $H(e^{j\omega})$
- Thus  $e^{j\omega n}$  is an eigen function of the system

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## The Frequency Response

- The quantity  $H(e^{j\omega})$  is called the **frequency response** of the LTI discrete-time system
- $H(e^{j\omega})$  provides a frequency-domain description of the system
- $H(e^{j\omega})$  is precisely the DTFT of the impulse response  $\{h[n]\}$  of the system

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## The Frequency Response

- $H(e^{j\omega})$ , in general, is a complex function of  $\omega$  with a period  $2\pi$
- It can be expressed in terms of its real and imaginary parts

$$H(e^{j\omega}) = H_{re}(e^{j\omega}) + j H_{im}(e^{j\omega})$$

or, in terms of its magnitude and phase,

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg H(e^{j\omega})$$

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## The Frequency Response

- The function  $|H(e^{j\omega})|$  is called the **magnitude response** and the function  $\theta(\omega)$  is called the **phase response** of the LTI discrete-time system
- Design specifications for the LTI discrete-time system, in many applications, are given in terms of the magnitude response or the phase response or both

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## The Frequency Response

- In some cases, the magnitude function is specified in **decibels** as

$$\mathcal{G}(\omega) = 20 \log_{10} |H(e^{j\omega})| \text{ dB}$$

where  $\mathcal{G}(\omega)$  is called the **gain function**

- The negative of the gain function

$$\mathcal{A}(\omega) = -\mathcal{G}(\omega)$$

is called the **attenuation or loss function**

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## The Frequency Response

- **Note:** Magnitude and phase functions are real functions of  $\omega$ , whereas the frequency response is a complex function of  $\omega$
- If the impulse response  $h[n]$  is real then it follows from Table 3.2 that the magnitude function is an even function of  $\omega$ :

$$|H(e^{j\omega})| = |H(e^{-j\omega})|$$

and the phase function is an odd function of  $\omega$ :

$$\theta(\omega) = -\theta(-\omega)$$

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## The Frequency Response

- Likewise, for a real impulse response  $h[n]$ ,  $H_{re}(e^{j\omega})$  is even and  $H_{im}(e^{j\omega})$  is odd
- **Example** - Consider the  $M$ -point moving average filter with an impulse response given by

$$h[n] = \begin{cases} 1/M, & 0 \leq n \leq M-1 \\ 0, & \text{otherwise} \end{cases}$$

- Its frequency response is then given by

$$H(e^{j\omega}) = \frac{1}{M} \sum_{n=0}^{M-1} e^{-j\omega n}$$

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## The Frequency Response

- Likewise, for a real impulse response  $h[n]$ ,  $H_{re}(e^{j\omega})$  is even and  $H_{im}(e^{j\omega})$  is odd
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- Its frequency response is then given by

$$H(e^{j\omega}) = \frac{1}{M} \sum_{n=0}^{M-1} e^{-j\omega n}$$

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## The Frequency Response

- Or,  $H(e^{j\omega}) = \frac{1}{M} \left( \sum_{n=0}^{\infty} e^{-j\omega n} - \sum_{n=M}^{\infty} e^{-j\omega n} \right)$
- $$= \frac{1}{M} \left( \sum_{n=0}^{\infty} e^{-j\omega n} \right) (1 - e^{-jM\omega}) = \frac{1}{M} \cdot \frac{1 - e^{-jM\omega}}{1 - e^{-j\omega}}$$
- $$= \frac{1}{M} \cdot \frac{\sin(M\omega/2)}{\sin(\omega/2)} e^{-j(M-1)\omega/2}$$

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## The Frequency Response

- Thus, the magnitude response of the  $M$ -point moving average filter is given by

$$|H(e^{j\omega})| = \left| \frac{1}{M} \cdot \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right|$$

and the phase response is given by

$$\theta(\omega) = -\frac{(M-1)\omega}{2} + \pi \sum_{k=0}^{\lfloor M/2 \rfloor} \mu \left( \omega - \frac{2\pi k}{M} \right)$$

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## Frequency Response Computation Using MATLAB

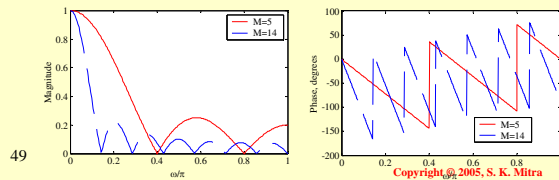
- The function `freqz(h, 1, w)` can be used to determine the values of the frequency response vector  $h$  at a set of given frequency points  $w$
- From  $h$ , the real and imaginary parts can be computed using the functions `real` and `imag`, and the magnitude and phase functions using the functions `abs` and `angle`

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## Frequency Response Computation Using MATLAB

- **Example** - Program 3\_2.m can be used to generate the magnitude and gain responses of an  $M$ -point moving average filter as shown below



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## Frequency Response Computation Using MATLAB

- The phase response of a discrete-time system when determined by a computer may exhibit jumps by an amount  $2\pi$  caused by the way the arctangent function is computed
- The phase response can be made a continuous function of  $\omega$  by unwrapping the phase response across the jumps

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## Frequency Response Computation Using MATLAB

- To this end the function `unwrap` can be used, provided the computed phase is in radians
- The jumps by the amount of  $2\pi$  should not be confused with the jumps caused by the zeros of the frequency response as indicated in the phase response of the moving average filter

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## Steady-State Response

- Note that the frequency response also determines the steady-state response of an LTI discrete-time system to a sinusoidal input
- **Example** - Determine the steady-state output  $y[n]$  of a real coefficient LTI discrete-time system with a frequency response  $H(e^{j\omega})$  for an input

$$x[n] = A \cos(\omega_o n + \phi), \quad -\infty < n < \infty$$

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## Steady-State Response

- We can express the input  $x[n]$  as

$$x[n] = g[n] + g^*[n]$$

where

$$g[n] = \frac{1}{2} A e^{j\phi} e^{j\omega_o n}$$

- Now the output of the system for an input  $e^{j\omega_o n}$  is simply

$$H(e^{j\omega_o}) e^{j\omega_o n}$$

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## Steady-State Response

- Because of linearity, the response  $v[n]$  to an input  $g[n]$  is given by

$$v[n] = \frac{1}{2} A e^{j\phi} H(e^{j\omega_o}) e^{j\omega_o n}$$

- Likewise, the output  $v^*[n]$  to the input  $g^*[n]$  is

$$v^*[n] = \frac{1}{2} A e^{-j\phi} H(e^{-j\omega_o}) e^{-j\omega_o n}$$

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## Steady-State Response

- Combining the last two equations we get

$$\begin{aligned} y[n] &= v[n] + v^*[n] \\ &= \frac{1}{2} A e^{j\phi} H(e^{j\omega_o}) e^{j\omega_o n} + \frac{1}{2} A e^{-j\phi} H(e^{-j\omega_o}) e^{-j\omega_o n} \\ &= \frac{1}{2} A |H(e^{j\omega_o})| \{ e^{j\theta(\omega_o)} e^{j\phi} e^{j\omega_o n} + e^{-j\theta(\omega_o)} e^{-j\phi} e^{-j\omega_o n} \} \\ &= A |H(e^{j\omega_o})| \cos(\omega_o n + \theta(\omega_o) + \phi) \end{aligned}$$

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## Steady-State Response

- Thus, the output  $y[n]$  has the same sinusoidal waveform as the input with two differences:
  - the amplitude is multiplied by  $|H(e^{j\omega_o})|$ , the value of the magnitude function at  $\omega = \omega_o$
  - the output has a **phase lag** relative to the input by an amount  $\theta(\omega_o)$ , the value of the phase function at  $\omega = \omega_o$

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## Response to a Causal Exponential Sequence

- The expression for the steady-state response developed earlier assumes that the system is initially relaxed before the application of the input  $x[n]$
- In practice, excitation  $x[n]$  to a discrete-time system is usually a right-sided sequence applied at some sample index  $n = n_o$
- We develop the expression for the output for such an input

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## Response to a Causal Exponential Sequence

- Without any loss of generality, assume  $x[n] = 0$  for  $n < 0$

- From the input-output relation

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

we observe that for an input

$$x[n] = e^{j\omega n} \mu[n]$$

the output is given by

$$y[n] = \left( \sum_{k=0}^n h[k] e^{j\omega(n-k)} \right) \mu[n]$$

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## Response to a Causal Exponential Sequence

- Or,  $y[n] = \left( \sum_{k=0}^n h[k] e^{-j\omega k} \right) e^{j\omega n} \mu[n]$
- The output for  $n < 0$  is  $y[n] = 0$
- The output for  $n \geq 0$  is given by
 
$$y[n] = \left( \sum_{k=0}^n h[k] e^{-j\omega k} \right) e^{j\omega n}$$

$$= \left( \sum_{k=0}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n} - \left( \sum_{k=n+1}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

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## Response to a Causal Exponential Sequence

- Or,
 
$$y[n] = H(e^{j\omega}) e^{j\omega n} - \left( \sum_{k=n+1}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$
- The first term on the RHS is the same as that obtained when the input is applied at  $n = 0$  to an initially relaxed system and is the **steady-state response**:

$$y_{sr}[n] = H(e^{j\omega}) e^{j\omega n}$$

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## Response to a Causal Exponential Sequence

- The second term on the RHS is called the **transient response**:

$$y_{tr}[n] = - \left( \sum_{k=n+1}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

- To determine the effect of the above term on the total output response, we observe

$$|y_{tr}[n]| = \left| \sum_{k=n+1}^{\infty} h[k] e^{-j\omega(k-n)} \right| \leq \sum_{k=n+1}^{\infty} |h[k]| \leq \sum_{k=0}^{\infty} |h[k]|$$

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## Response to a Causal Exponential Sequence

- For a causal, stable LTI IIR discrete-time system,  $h[n]$  is absolutely summable
- As a result, the transient response  $y_{tr}[n]$  is a bounded sequence

- Moreover, as  $n \rightarrow \infty$ ,

$$\sum_{k=n+1}^{\infty} |h[k]| \rightarrow 0$$

and hence, the transient response decays to zero as  $n$  gets very large

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## Response to a Causal Exponential Sequence

- For a causal FIR LTI discrete-time system with an impulse response  $h[n]$  of length  $N + 1$ ,  $h[n] = 0$  for  $n > N$
- Hence,  $y_{tr}[n] = 0$  for  $n > N - 1$
- Here the output reaches the steady-state value  $y_{sr}[n] = H(e^{j\omega})e^{j\omega n}$  at  $n = N$

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## The Concept of Filtering

- One application of an LTI discrete-time system is to **pass** certain frequency components in an input sequence without any distortion (if possible) and to **block** other frequency components
- Such systems are called digital filters and one of the main subjects of discussion in this course

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## The Concept of Filtering

- The key to the filtering process is

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- It expresses an arbitrary input as a linear weighted sum of an infinite number of exponential sequences, or equivalently, as a linear weighted sum of sinusoidal sequences

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## The Concept of Filtering

- Thus, by appropriately choosing the values of the magnitude function  $|H(e^{j\omega})|$  of the LTI digital filter at frequencies corresponding to the frequencies of the sinusoidal components of the input, some of these components can be selectively heavily attenuated or filtered with respect to the others

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## The Concept of Filtering

- To understand the mechanism behind the design of frequency-selective filters, consider a real-coefficient LTI discrete-time system characterized by a magnitude function

$$|H(e^{j\omega})| \cong \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

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## The Concept of Filtering

- We apply an input

$$x[n] = A \cos \omega_1 n + B \cos \omega_2 n, \quad 0 < \omega_1 < \omega_c < \omega_2 < \pi$$

to this system

- Because of linearity, the output of this system is of the form

$$y[n] = A |H(e^{j\omega_1})| \cos(\omega_1 n + \theta(\omega_1)) \\ + B |H(e^{j\omega_2})| \cos(\omega_2 n + \theta(\omega_2))$$

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## The Concept of Filtering

- As

$$|H(e^{j\omega_1})| \cong 1, \quad |H(e^{j\omega_2})| \cong 0$$

the output reduces to

$$y[n] \cong A |H(e^{j\omega_1})| \cos(\omega_1 n + \theta(\omega_1))$$

- Thus, the system acts like a **lowpass filter**
- In the following example, we consider the design of a very simple digital filter

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## The Concept of Filtering

- Example** - The input consists of a sum of two sinusoidal sequences of angular frequencies 0.1 rad/sample and 0.4 rad/sample
- We need to design a highpass filter that will pass the high-frequency component of the input but block the low-frequency component
- For simplicity, assume the filter to be an FIR filter of length 3 with an impulse response:

$$h[0] = h[2] = \alpha, \quad h[1] = \beta$$

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## The Concept of Filtering

- The convolution sum description of this filter is then given by

$$y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] \\ = \alpha x[n] + \beta x[n-1] + \alpha x[n-2]$$

- $y[n]$  and  $x[n]$  are, respectively, the output and the input sequences
- Design Objective:** Choose suitable values of  $\alpha$  and  $\beta$  so that the output is a sinusoidal sequence with a frequency 0.4 rad/sample

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## The Concept of Filtering

- Now, the frequency response of the FIR filter is given by

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} \\ = \alpha(1 + e^{-j2\omega}) + \beta e^{-j\omega} \\ = 2\alpha \left( \frac{e^{j\omega} + e^{-j\omega}}{2} \right) e^{-j\omega} + \beta e^{-j\omega} \\ = (2\alpha \cos \omega + \beta) e^{-j\omega}$$

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## The Concept of Filtering

- The magnitude and phase functions are

$$\begin{aligned} |H(e^{j\omega})| &= 2\alpha \cos \omega + \beta \\ \theta(\omega) &= -\omega \end{aligned}$$

- In order to block the low-frequency component, the magnitude function at  $\omega = 0.1$  should be equal to zero
- Likewise, to pass the high-frequency component, the magnitude function at  $\omega = 0.4$  should be equal to one

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## The Concept of Filtering

- Thus, the two conditions that must be satisfied are

$$|H(e^{j0.1})| = 2\alpha \cos(0.1) + \beta = 0$$

$$|H(e^{j0.4})| = 2\alpha \cos(0.4) + \beta = 1$$

- Solving the above two equations we get

$$\alpha = -6.76195$$

$$\beta = 13.456335$$

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## The Concept of Filtering

- Thus the output-input relation of the FIR filter is given by

$$y[n] = -6.76195(x[n] + x[n-2]) + 13.456335x[n-1]$$

where the input is

$$x[n] = \{\cos(0.1n) + \cos(0.4n)\} \mu[n]$$

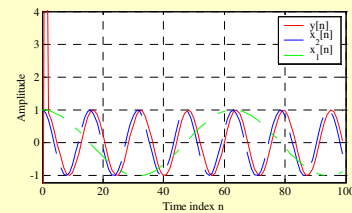
- Program 3\_3.m can be used to verify the filtering action of the above system

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## The Concept of Filtering

- Figure below shows the plots generated by running this program



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## The Concept of Filtering

- The first seven samples of the output are shown below

$n$	$\cos(0.1n)$	$\cos(0.4n)$	$x[n]$	$y[n]$
0	1.0	1.0	2.0	-13.52390
1	0.9950041	0.9210609	1.9160652	13.956333
2	0.9800665	0.6967067	1.6767733	0.9210616
3	0.9553364	0.3623577	1.3176942	0.6967064
4	0.9210609	-0.0291995	0.8918614	0.3623572
5	0.8775825	-0.4161468	0.4614357	-0.0292002
6	0.8253356	-0.7373937	0.0879419	-0.4161467

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## The Concept of Filtering

- From this table, it can be seen that, neglecting the least significant digit,

$$y[n] = \cos(0.4(n-1)) \quad \text{for } n \geq 2$$

- Computation of the present value of the output requires the knowledge of the present and two previous input samples
- Hence, the first two output samples,  $y[0]$  and  $y[1]$ , are the result of assumed zero input sample values at  $n = -1$  and  $n = -2$

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## The Concept of Filtering

- Therefore, first two output samples constitute the transient part of the output
- Since the impulse response is of length 3, the steady-state is reached at  $n = N = 2$
- Note also that the output is delayed version of the high-frequency component  $\cos(0.4n)$  of the input, and the delay is one sample period

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## Phase Delay

- If the input  $x[n]$  to an LTI system  $H(e^{j\omega})$  is a sinusoidal signal of frequency  $\omega_o$ :
- $$x[n] = A \cos(\omega_o n + \phi), \quad -\infty < n < \infty$$
- Then, the output  $y[n]$  is also a sinusoidal signal of the same frequency  $\omega_o$  but lagging in phase by  $\theta(\omega_o)$  radians:

$$y[n] = A |H(e^{j\omega_o})| \cos(\omega_o n + \theta(\omega_o) + \phi), \quad -\infty < n < \infty$$

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## Phase Delay

- We can rewrite the output expression as
- $$y[n] = A |H(e^{j\omega_o})| \cos(\omega_o (n - \tau_p(\omega_o) + \phi))$$
- where
- $$\tau_p(\omega_o) = -\frac{\theta(\omega_o)}{\omega_o}$$
- is called the phase delay
- The minus sign in front indicates phase lag

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## Phase Delay

- Thus, the output  $y[n]$  is a time-delayed version of the input  $x[n]$
  - In general,  $y[n]$  will not be delayed replica of  $x[n]$  unless the phase delay  $\tau_p(\omega_o)$  is an integer
- ➔ Phase delay has a physical meaning only with respect to the underlying continuous-time functions associated with  $y[n]$  and  $x[n]$

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## Group Delay

- When the input is composed of many sinusoidal components with different frequencies that are not harmonically related, each component will go through different phase delays
- In this case, the signal delay is determined using the group delay defined by

$$\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}$$

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## Group Delay

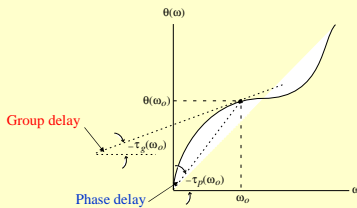
- In defining the group delay, it is assumed that the phase function is unwrapped so that its derivatives exist
- Group delay also has a physical meaning only with respect to the underlying continuous-time functions associated with  $y[n]$  and  $x[n]$

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## Phase and Group Delays

- A graphical comparison of the two types of delays are indicated below



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## Phase and Group Delays

- Example** - The phase function of the FIR filter  $y[n] = \alpha x[n] + \beta x[n-1] + \alpha x[n-2]$  is  $\theta(\omega) = -\omega$
- Hence its group delay is given by  $\tau_g(\omega) = 1$  verifying the result obtained earlier by simulation

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## Phase and Group Delays

- Example** - For the  $M$ -point moving-average filter

$$h[n] = \begin{cases} 1/M, & 0 \leq n \leq M-1 \\ 0, & \text{otherwise} \end{cases}$$

the phase function is

$$\theta(\omega) = -\frac{(M-1)\omega}{2} + \pi \sum_{k=0}^{\lfloor M/2 \rfloor} \mu \left( \omega - \frac{2\pi k}{M} \right)$$

- Hence its group delay is

$$\tau_g(\omega) = \frac{M-1}{2}$$

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## Phase and Group Delays

- Physical significance of the two delays are better understood by examining the continuous-time case
- Consider an LTI continuous-time system with a frequency response

$$H_a(j\Omega) = |H_a(j\Omega)| e^{j\theta_a(\Omega)}$$

and excited by a narrow-band amplitude modulated continuous-time signal

$$x_a(t) = a(t) \cos(\Omega_c t)$$

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## Phase and Group Delays

- $a(t)$  is a lowpass modulating signal with a band-limited continuous-time Fourier transform given by

$$|A(j\Omega)| = 0, \quad |\Omega| > \Omega_o$$

and  $\cos(\Omega_c t)$  is the high-frequency carrier signal

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## Phase and Group Delays

- We assume that in the frequency range  $\Omega_c - \Omega_o < |\Omega| < \Omega_c + \Omega_o$  the frequency response of the continuous-time system has a constant magnitude and a linear phase:

$$\begin{aligned} |H_a(j\Omega)| &= |H_a(j\Omega_c)| \\ \theta_a(\Omega) &= \theta_a(\Omega_c) - (\Omega - \Omega_c) \frac{d\theta_a(\Omega)}{d\Omega} \bigg|_{\Omega=\Omega_c} \\ &= -\Omega_c \tau_p(\Omega_c) + (\Omega - \Omega_c) \tau_g(\Omega_c) \end{aligned}$$

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## Phase and Group Delays

- Now, the CTFT of  $x_a(t)$  is given by

$$X_a(j\Omega) = \frac{1}{2}(A(j[\Omega + \Omega_c]) + A(j[\Omega - \Omega_c]))$$

- Also, because of the band-limiting constraint  $X_a(j\Omega) = 0$  outside the frequency range  $\Omega_c - \Omega_o < |\Omega| < \Omega_c + \Omega_o$

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## Phase and Group Delays

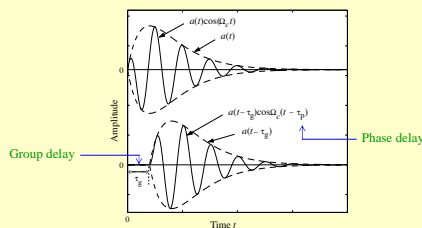
- As a result, the output response  $y_a(t)$  of the LTI continuous-time system is given by  $y_a(t) = a(t - \tau_g(\Omega_c))\cos\Omega_c(t - \tau_p(\Omega_c))$  assuming  $|H_a(j\Omega_c)| = 1$
- As can be seen from the above equation, the group delay  $\tau_g(\Omega_c)$  is precisely the delay of the envelope  $a(t)$  of the input signal  $x_a(t)$ , whereas, the phase delay  $\tau_p(\Omega_c)$  is the delay of the carrier

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## Phase and Group Delays

- The figure below illustrates the effects of the two delays on an amplitude modulated sinusoidal signal



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## Phase and Group Delays

- The waveform of the underlying continuous-time output shows distortion when the group delay is not constant over the bandwidth of the modulated signal
- If the distortion is unacceptable, an allpass delay equalizer is usually cascaded with the LTI system so that the overall group delay is approximately linear over the frequency range of interest while keeping the magnitude response of the original LTI system unchanged

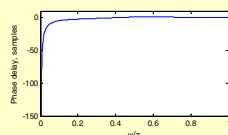
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## Phase Delay Computation Using MATLAB

- Phase delay can be computed using the function `phasedelay`
- Figure below shows the phase delay of the DTFT

$$H(e^{j\omega}) = \frac{0.1367(1 - e^{-j2\omega})}{1 - 0.5335e^{-j\omega} + 0.7265e^{-j2\omega}}$$



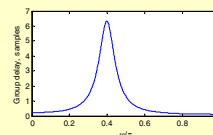
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## Group Delay Computation Using MATLAB

- Group delay can be computed using the function `grpdelay`
- Figure below shows the group delay of the DTFT

$$H(e^{j\omega}) = \frac{0.1367(1 - e^{-j2\omega})}{1 - 0.5335e^{-j\omega} + 0.7265e^{-j2\omega}}$$



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