Chapter 3 Kirchhoff's laws, network equations, and introduction to network functions

By combining circuit elements in a proper manner, a circuit can be constructed to represent the behavior of an actual device. For example, under certain conditions, the field coil of a generator can be represented as a combination of R and L elements placed end to end as shown in Fig. 3-1.

Referring to the field coil of a generator and its equivalent circuit as shown in Fig. 3-1, the two terminals a and b represent actual terminals

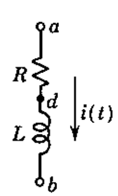

Fig. 3-1 Series connection of two elements.

of the field coil, but the junction between the two elements (point d) does not correspond to any particular point in the field coil. This is not surprising since, according to our definitions of the circuit elements, R accounts for the dissipation of energy into heat, and L accounts for the storage of energy in the magnetic field, whenever a current flows between terminals a-b of the actual field coil. In the actual device the "resistance" and the

"inductance" are not separate "parts," but in the equivalent circuit we choose to represent them as such.

The only true similarity between the response of the actual device and its equivalent circuit is in the correspondence between their voltage-current characteristic at the two terminals a-b. What goes on inside the equivalent circuit may have no counterpart in the actual device. For example, as a result of the flow of current from a to b, a voltage will develop across R (Fig. 3-1). This voltage does not correspond to any voltage which can be *measured* in the actual field coil. On the other hand, voltage and current at terminals a-b of the field coil and of its equivalent circuit will be identical. We may consider the circuit shown in Fig. 3-1 to be a terminal pair a-b representing the actual device. In such a case the terminals a and b are called *accessible* terminals, and the terminal d is called an *inaccessible* terminal.

3-1 Series and parallel connections of elements

Elements connected end to end, such that they carry the *same current*, are said to be in *series*. In Fig. 3-1, R and L are connected in series. If at any instant of time there is a current $i_{ad}(t)$ in R, there will also be a current $i_{db}(t)$ such that $i_{ad}(t) \equiv i_{db}(t) \equiv i$. This result is arrived at from the fundamental assumption that in a terminal pair the charges which enter one terminal must come out from the other terminal. This is called the "assumption of continuity of current."

Elements connected between a pair of terminals are said to be connected in "parallel." Elements connected in parallel will have the *same voltage* across them.

The connection of the elements R and C in Fig. 3-2 is an example of *parallel connection* of two elements. In this circuit the two elements are

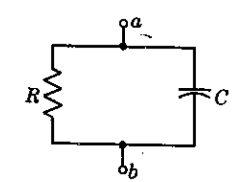

Fig. 3-2 Parallel connection of two elements.

placed so that the same voltage will always exist across them. Although this circuit may correspond to the actual connection of two physical devices, it may also be the equivalent circuit for many practical devices. In this circuit terminals a-b may represent the input terminals of an electronic amplifier. In that case the capacitance C may not be placed in the circuit intentionally, but may represent the combination of certain unavoidable features of the electronic device. As another example, the same circuit may, with respect to the voltage-current characteristic at terminals a-b, represent the input to a telephone cable.

The circuit shown in Fig. 3-3*a* is an example of a "series-parallel" connection of circuit elements. The series connection of R and L is connected in parallel with C. It is interesting to note that this circuit might represent a coil of wire. The resistance and inductance represent the same types of energy conversion as explained in connection with Fig. 3-1. The capacitance C would account for the electric-energy storage in the electric field between the windings of the coil.

The circuit of Fig. 3-3*b* is another example of a series-parallel circuit. In this circuit the resistance R_1 is in series with the parallel combination

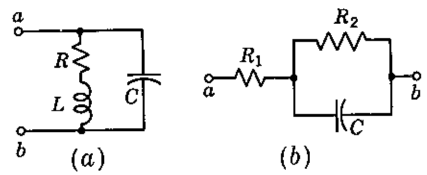

Fig. 3-3 Series-parallel connection of elements.

of R_2 and C. This circuit, like all the others, may represent the actual combination of several nearly "pure" elements or may be the equivalent circuit of an actual device or a portion of such a device. (Figure 3-3*b* may represent the input circuit of an oscilloscope probe.)

3-2 Network terminology

The representation of electrical systems frequently requires the combination of many sources and elements connected in a more complicated manner than the series or parallel arrangement described above. To facilitate discussion and analysis of networks the special terminology given below is used.

NETWORK An interconnection of circuit elements is termed a network, or circuit. A network may contain both active and passive elements or may consist of passive elements only. In the former case it is termed an *active* network, and in the latter case the term *passive*, or *source-free*, applies. If the passive elements in a network are all of the same type, i.e., all resistances or all inductances or all capacitances, the network is termed a resistive, inductive, or capacitive network, respectively.

RESPONSES OF A NETWORK The waveform, or function, representing the current in or voltage across a passive element or a combination of elements, as well as the voltage across an ideal current source and the current in ideal voltage sources, is termed a *response of a network*. Thus all voltage and current waveforms which are not specified through the ideal sources in a network are responses of the network.

NODES We shall call a point in the network common to two or more elements a node. For example, in Fig. 3-4, the points a, b, d, f, g, h, k, m, and n are nodes.

JUNCTIONS We shall call a node common to three or more elements a junction. In Fig. 3-4 the nodes a, b, d, and m are junctions, whereas the nodes g, h, k, and n are not junctions, since each of them is common to two elements only.

BRANCH We shall call a single element or a series connection of elements between any two junctions a branch. In Fig. 3-4 the following branches are identified: branch ab, consisting of the current source $i(t)$; the resistive branch ab, consisting of R_1; branch ad, consisting of C_1; branch db, consisting

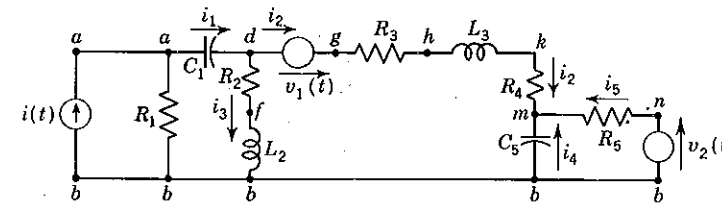

Fig. 3-4 Diagram of a network.

of R_2 and L_2; branch dm, consisting of $v_1(t)$, R_3, L_3, and R_4; branch mb, consisting of C_5; branch mnb, consisting of R_5 and $v_2(t)$. Note that, in accordance with our definition, in Fig. 3-4 km, hkm, and df are not branches since they do not connect two junctions.

PASSIVE AND ACTIVE BRANCHES If a branch contains no sources, it is called a passive branch; otherwise it is an active branch. In Fig. 3-4 the branch mnb is an active branch, and the branch dfb is a passive branch. The current source $i(t)$ in this figure constitutes a branch by itself.

BRANCH CURRENT A current flowing in a branch is called a branch current. The same branch current flows through all the elements in a given branch, since the elements in a branch are in series.

LOOP We shall call any closed path through the circuit elements of a network a loop. In Fig. 3-4 the following loops are identified: loop a-b-a, consisting of R_1 and $i(t)$; loop a-d-f-b-a, consisting of C_1, R_2, L_2, and R_1; loop a-b-f-d-a, consisting of $i(t)$, L_2, R_2, and C_1; loop d-g-h-k-m-n-b-f-d, consisting of $v_1(t)$, R_3, L_3, R_4, R_5, $v_2(t)$, L_2, and R_2. The reader may trace a few more loops as an exercise. If a loop contains one element of a branch, it will contain all the elements of that branch. In Fig. 3-4 the loop a-b-f-d-a, which contains R_2, also contains L_2, which with R_2 forms the branch dfb. In Fig. 3-4 loop a-b-a has two branches, that is, $i(t)$ and R_1; and loop a-b-f-d-a has three branches, R_1, C_1, and dfb. A branch may be common to more than two loops. In Fig. 3-4 the branch R_1 is common to loops a-d-f-b-a, a-d-g-h-k-m-b-a, and others.

MESH In a given diagram of a network a loop which does not encircle or enclose another loop and cannot be divided into other loops is called a mesh. In Fig. 3-4 the loop *m-n-b-m* is a mesh, whereas the loop *a-d-g-h-k-m-b-a* is not a mesh, since it encloses the loop *a-d-f-b-a*. A branch may not be common to more than two meshes. The reasons for defining meshes as well as loops are discussed in Chap. 11.

3-3 Restrictions on variables of networks: Kirchhoff's laws

As a result of the interconnection of elements in a network, certain restrictions (*constraints*) are placed on the currents and voltages associated with these elements. For example, in the series connection of the R and L elements in Fig. 3-1, the principle of continuity of current requires that $i_{ad} \equiv i_{db}$. In the parallel connection of the R and C elements in Fig. 3-2, the definition of the voltage requires that the same voltages exist across R and C. Therefore, if the current i_{ad} in Fig. 3-1 or the voltage across R in Fig. 3-2 is specified, the current i_{db} in Fig. 3-1 or the voltage across C in Fig. 3-2, respectively, will also be specified. *Thus, when we connect elements in series, we place a restriction on the current through them. When we connect elements in parallel, a restriction is established on the voltage across them.*

The application of the principles of continuity of current and the law of conservation of energy establishes certain additional restrictions on the currents and voltages associated with elements in a network. We shall first state these restrictions and then deduce them from the above principles. The treatment of circuit problems can be approached by stating these restrictions as "laws" of circuits. These laws are called, after the physicist Gustav Robert Kirchhoff (1824–1887), Kirchhoff's laws.

> *Kirchhoff's voltage law* At any instant the sum of the voltages around any loop is identically zero.

As an illustration, we apply this law to the loop *a-d-f-b-a* in Fig. 3-4:

$$v_{ad} + v_{df} + v_{fb} + v_{ba} = 0$$

Similarly, for the loop *a-d-g-h-k-m-n-b-a*,

$$v_{ad} + v_{dg} + v_{gh} + v_{hk} + v_{km} + v_{mn} + v_{nb} + v_{ba} = 0$$

To state Kirchhoff's current law in compact form, we shall give two definitions in connection with reference arrows for currents.

If the head of the reference arrow of a current points toward (or away from) a node, we say that the current is entering (or leaving) that node. If a current i enters a node through an element, a current $-i$ leaves the node through that element.

Using these definitions, Kirchhoff's current law is stated as follows:

> *Kirchhoff's current law* At any instant the algebraic sum of the currents *entering* a node is identically zero, or at any instant the sum of the currents *leaving* a node is identically zero.

As an illustration of this law, consider the node m in Fig. 3-4, where $i_2 + i_4 + i_5 = 0$. The reference arrows of the three currents point toward the junction m, and therefore i_2, i_4, and i_5 enter this junction, and their sum is zero. At the junction d, i_2 and i_3 both leave d, but i_1 enters d. If i_1 enters d, then $-i_1$ leaves d, and the application of Kirchhoff's current law gives $-i_1 + i_2 + i_3 = 0$. The latter equation may be written as

$$i_1 = i_2 + i_3$$

which can be interpreted as: The sum of currents whose reference arrows enter a node (i_1 entering d) is equal to the sum of currents whose reference arrows leave that node (i_2 and i_3 leaving d).

Notice that, in the branch *dghkm* of Fig. 3-4, i_2 is the branch current and flows through all the elements of that branch. Thus

$$i_{dg} = i_{gh} = i_{hk} = i_{km} = i_2$$

The equation $i_{dg} = i_{gh}$ can also be considered an expression of Kirchhoff's current law since i_{dg} is the current entering node g, and i_{gh} is the current leaving node g.

In discussing the series connection of elements, we referred to the assumption of continuity of current. The application of this principle to the junction of elements results in Kirchhoff's current law. If the sum of the currents entering a junction were not equal to the sum of the currents leaving that junction, there would be an accumulation of charge at that junction. The effect of accumulation of charge is represented by the circuit element capacitance, and accumulation of charge at a terminal has no meaning in terms of the concepts already defined.

Example 3-1 Let i_1 and i_2 in Fig. 3-5 be given by $i_1 = 8u(t)$ and $i_2 = 3tu(t)$. With these two currents specified, we are no longer at liberty to specify $i_3(t)$ since $i_3(t) = i_1 + i_2 = (8 + 3t)u(t)$.

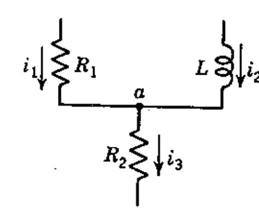

Fig. 3-5 Illustration of Kirchhoff's current law: $i_1 + i_2 = i_3$.

At time t the amount of charge which has entered the junction a is $\int_{-\infty}^{t} (i_1 + i_2)\,d\tau$, and the amount of charge which has left a is $\int_{-\infty}^{t} i_3\,d\tau$. If $i_3 \neq i_1 + i_2$, charge equal to $\int_{-\infty}^{t} (i_1 + i_2 - i_3)\,d\tau$ has accumulated at a. As mentioned above, this is contrary to our concept of terminal pairs, and therefore $i_1 + i_2 = i_3$.

3-4 Derivation of Kirchhoff's voltage law from the current law and the law of conservation of energy

In Fig. 3-6*a* a closed path is shown. Let us assume that the terminal pairs shown in this figure are passive (do not include sources) and that the only source in the path is $v(t)$.

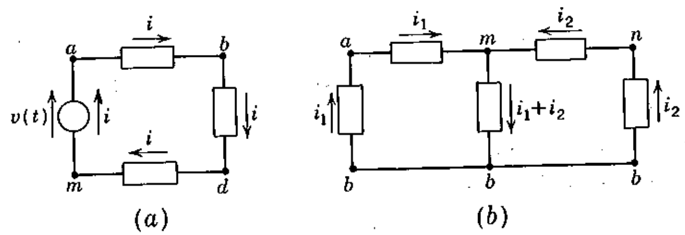

Fig. 3-6 (a) Network consisting of one loop. (b) A two-mesh network.

At any instant, the power input to the terminal pairs *a-b*, *b-d*, and *d-m* is $v_{ab}i_{ab}$, $v_{bd}i_{bd}$, and $v_{dm}i_{dm}$, respectively. Since $i_{ab} = i_{bd} = i_{dm} = i$, the rate of delivery of energy to the passive elements is

$$v_{ab}i_{ab} + v_{bd}i_{bd} + v_{dm}i_{dm} = i(v_{ab} + v_{bd} + v_{dm})$$

The source delivers energy at the rate iv_{am}, and therefore

$$iv_{am} = i(v_{ab} + v_{bd} + v_{dm})$$

so that

$$v_{am} = v_{ab} + v_{bd} + v_{dm} = -v_{ma} \tag{3-1}$$

or

$$v_{ab} + v_{bd} + v_{dm} + v_{ma} = 0 \tag{3-2}$$

which is the statement of Kirchhoff's voltage law for this circuit.

Most often, when the circuit contains a source in series with passive terminal pairs, it is more convenient to express Kirchhoff's voltage law in the form of Eq. (3-1), namely, with the source function on one side of the equation, rather than in the form of Eq. (3-2).

The above derivation was made in the special case of a network consisting of one loop. The argument can be extended to apply to any network, regardless of its geometry. In Fig. 3-6*b* a two-mesh network, without any source, is shown. In conformity with Kirchhoff's current law, the current in branch *mb* of Fig. 3-6*b* is $i_1 + i_2$.

By the law of conservation of energy the total power received by all the terminal pairs is zero. (This means that at any given instant some of the terminal pairs receive and others deliver energy.)

$$-v_{ab}i_1 + v_{am}i_1 + v_{mb}(i_1 + i_2) - v_{mn}i_2 - v_{nb}i_2 = 0$$
$$i_1(v_{am} + v_{mb} - v_{ab}) + i_2(v_{mb} - v_{mn} - v_{nb}) = 0 \tag{3-2a}$$

In the network of Fig. 3-6*b* the waveform of i_2 can be changed without any change in the waveform of i_1. This can be achieved by changing the elements in the terminal pairs of the network (for example, terminal pairs *m-n*, *n-b*, and *m-b*). In other words, the only restriction placed by the network on the waveform of the current is that given by $i_{mb} = i_1 + i_2$. This is a restriction on the waveform of i_{mb}. One of the currents i_1 and i_2 can be specified independently of the other.

Returning now to Eq. (3-2*a*), we note that the form of this equation is

$$i_1(t)f_1(t) + i_2(t)f_2(t) = 0 \tag{3-2b}$$

Now, since $i_1(t)$ or $i_2(t)$ can be specified arbitrarily, it follows that $f_1(t) \equiv 0$ and $f_2(t) \equiv 0$. To clarify this argument, let us assume that $i_2(t) \equiv 0$ and $i_1(t) \not\equiv 0$. Then $f_1(t) \equiv 0$. If, on the other hand, we let $i_1(t) \equiv 0$ and choose $i_2(t) \not\equiv 0$, then $f_2(t) \equiv 0$. Hence, in Eq. (3-2*a*), $v_{am} + v_{mb} - v_{ab} = 0$, and $v_{mb} - v_{mn} - v_{nb} = 0$.

These equations correspond to Kirchhoff's voltage law for each of the two meshes *a-m-b-a* and *m-n-b-m*.

Although the derivation of Kirchhoff's voltage law has dealt only with particular examples (Fig. 3-6*a* and *b*), the result is general and can be proved generally by application of a similar procedure.

3-5 The application of Kirchhoff's laws

We shall now show how the application of Kirchhoff's laws leads to integro-differential or differential equations for network responses. Before generalizing, two examples are appropriate.

Example 3-2 Apply Kirchhoff's voltage law to the circuit shown in Fig. 3-7, and show that the equation for $i(t)$ is an integrodifferential equation.

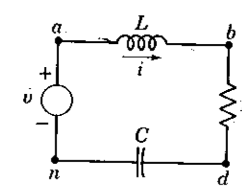

Fig. 3-7 A series R-L-C circuit excited by an ideal voltage source.

Solution Kirchhoff's voltage law applied to the single-loop circuit of Fig. 3-7 yields the equation

$$v_{ab} + v_{bd} + v_{dn} + v_{na} = 0$$

or since $v = v_{an} = -v_{na}$,

$$v_{ab} + v_{bd} + v_{dn} = v \tag{3-3}$$

We observe that the three voltage variables v_{ab}, v_{bd}, and v_{dn} can all be expressed in terms of the current i through the voltage-current relations for the L, R, and C elements; that is,

$$v_{ab} = L\frac{di}{dt} \qquad v_{bd} = Ri \qquad v_{dn} = \frac{1}{C}\int_{-\infty}^{t} i\,d\tau$$

Hence, using the above relations in Eq. (3-3),

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{t} i\,d\tau = v \tag{3-4}$$

Equation (3-4) is the required integrodifferential equation.

Example 3-3 Apply Kirchhoff's current law to the circuit shown in Fig. 3-8, and deduce the integrodifferential equation for $v(t)$.

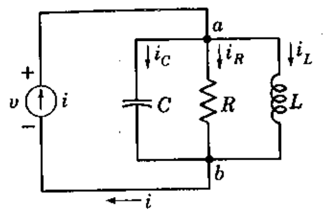

Fig. 3-8 A parallel R-L-C circuit excited by an ideal current source.

Solution Kirchhoff's current laws applied at junction b of Fig. 3-8 yields the equation

$$i_C + i_R + i_L - i = 0$$

or

$$i_C + i_R + i_L = i \tag{3-5}$$

We observe that the three current variables i_C, i_R, and i_L can all be expressed in terms of the same voltage $v = v_{ab}$ through the current-voltage relations for the elements C, R, and L; that is,

$$i_C = C\frac{dv}{dt} \qquad i_R = \frac{1}{R}v \qquad i_L = \frac{1}{L}\int_{-\infty}^{t} v\,d\tau$$

Hence, using the above relations in Eq. (3-5),

$$C\frac{dv}{dt} + \frac{1}{R}v + \frac{1}{L}\int_{-\infty}^{t} v\,d\tau = i \tag{3-6}$$

Equation (3-6) is the required integrodifferential equation.

The solution of integrodifferential equations such as Eqs. (3-4) and (3-6) is the subject of other chapters. Examples 3-2 and 3-3 illustrate the technique for arriving at the equation relating a response to the source of a network and point out several general properties of Kirchhoff-law equations as follows:

1 Application of Kirchhoff's laws to network loops and junctions will generally result in integrodifferential equations.
2 Application of the voltage law to series elements is most conveniently done by expressing each element voltage in the series connection in terms of the common current.
3 Application of the current law to parallel elements is most conveniently carried out by expressing each element current of the parallel combination in terms of the common voltage.
4 Different networks can yield integrodifferential equations of the same mathematical form [compare Eqs. (3-4) and (3-6)]. Such analogies are useful, and are discussed further in Chap. 4.

At this point in the development we note that in *resistive networks* every voltage-current relationship is algebraic; hence the application of Kirchhoff's laws will result in algebraic rather than integrodifferential equations. Below we show that *inductive* and *capacitive* networks can also be described by algebraic equations. Finally, in Sec. 3-10, we show that the use of operational notation allows the treatment of general networks in algebraic form.

3-6 Resistive, inductive, and capacitive ladder networks

When the passive elements in a network are all of one type, the relationship between response and the sources can be obtained by algebraic means. In this section we show how such equilibrium relations can be obtained in ladder networks. Ladder networks consist alternately of series and parallel elements as shown in Fig. 3-9.

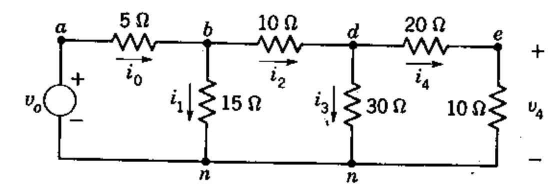

Fig. 3-9 Resistive ladder network.

Example 3-4 In the resistive ladder network of Fig. 3-9, deduce the relationship between v_4 and v_o, between i_4 and v_o, and between i_0 and v_o.

Solution In Fig. 3-9, a current with a reference arrow has been assigned to every branch. We now *begin* at the *output* v_4 and apply, in turn, the resistance voltage-current relationship, the voltage law, and the current law. At each step the variables are expressed in terms of v_4.

Thus, from the resistance voltage-current relation,

$$i_4 = \tfrac{1}{10}v_4 \qquad \text{and} \qquad v_{de} = 20i_4 = 2v_4 \tag{3-7}$$

Hence, from the voltage law,

$$v_{dn} = v_{de} + v_4 = 3v_4 \tag{3-8}$$

and from the resistance relation for the 30-ohm resistance,

$$i_3 = \frac{v_{dn}}{30} = \frac{v_4}{10} \tag{3-9}$$

From the current law,

$$i_2 = i_3 + i_4 = \frac{v_4}{10} + \frac{v_4}{10} = \frac{v_4}{5} \tag{3-10}$$

and from the resistance relation for R_{bd},

$$v_{bd} = 10i_2 = 2v_4 \tag{3-11}$$

From the voltage law,

$$v_{bn} = v_{bd} + v_{dn} = 5v_4 \tag{3-12}$$

From the resistance relation for the 15-ohm resistance,

$$i_1 = \tfrac{1}{15} v_{bn} = \tfrac{1}{3} v_4 \tag{3-13}$$

From the current law,

$$i_0 = i_1 + i_2 = \tfrac{1}{3} v_4 + \tfrac{1}{5} v_4 = \tfrac{8}{15} v_4 \tag{3-14}$$

From the resistance relation for R_{ab},

$$v_{ab} = 5 i_0 = \tfrac{8}{3} v_4 \tag{3-15}$$

and from the voltage law,

$$v_o = v_{ab} + v_{bn}$$
$$v_o = \tfrac{8}{3} v_4 + 5 v_4 = \tfrac{23}{3} v_4 \tag{3-16}$$

Equation (3-16) is one required result. Since $i_4 = v_4/10$, the relationship between i_4 and v_o is given by $v_o = \frac{23}{3}(10 i_4) = 230 i_4/3$. Finally, from Eq. (3-14),

$$v_4 = \tfrac{15}{8} i_0$$

so that

$$v_o = \tfrac{23}{3} \times \tfrac{15}{8} i_0 = \tfrac{115}{8} i_0 \tag{3-17}$$

which is also a required result.

Example 3-5 In the capacitance network of Fig. 3-10, deduce the relationship between i_3 and i, v_4 and i, and v_o and i.

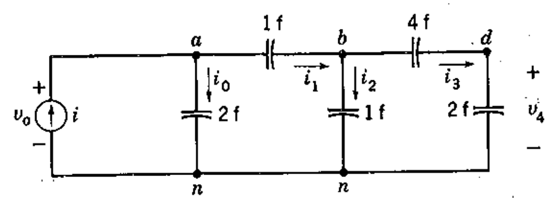

***Fig. 3-10* Capacitive ladder network.**

Solution The procedure which may be followed is completely analogous to that used in the resistive case if one uses

$$q_3 = \int_{-\infty}^{t} i_3 \, d\tau \qquad q_2 = \int_{-\infty}^{t} i_2 \, d\tau \qquad \cdots$$

We shall however use voltages and currents as variables and begin with

$$v_4 = \tfrac{1}{2} \int_{-\infty}^{t} i_3 \, d\tau = \tfrac{1}{2} q_3$$
$$v_{bd} = \tfrac{1}{4} \int_{-\infty}^{t} i_3 \, d\tau = \tfrac{1}{4} q_3$$

Hence, from the voltage law, $v_{bn} = v_3 + v_4$, or

$$v_{bn} = \tfrac{3}{4} \int_{-\infty}^{t} i_3 \, dt = \tfrac{3}{4} q_3$$

From the capacitance voltage-current relation,

$$i_2 = \frac{dv_{bn}}{dt} = \frac{3}{4} i_3$$

From the current law,

$$i_1 = i_2 + i_3 = \tfrac{7}{4} i_3 \tag{3-18}$$

The procedure should now be clear since it is analogous to the resistive case treated in Example 3-4. It is left as an excercise for the reader to show that the results are

$$i = \tfrac{27}{4} i_3 \tag{3-19}$$

$$i = \frac{27}{2} \frac{dv_4}{dt} \tag{3-20}$$

and

$$i = 2.7 \frac{dv_o}{dt} \tag{3-21}$$

The calculation of an inductive ladder is left as an exercise for the reader (Prob. 3-13).

3-7 Equivalent elements

We observe that in Example 3-4 the voltage-current relationship at the terminals of the source is as given by Eq. (3-17),

$$v_o = \tfrac{115}{8} i_0$$

so that at the source terminal the voltage-current relationship is the same as if a resistance of $\frac{115}{8}$ ohms were connected. Similarly, in Example 3-5,

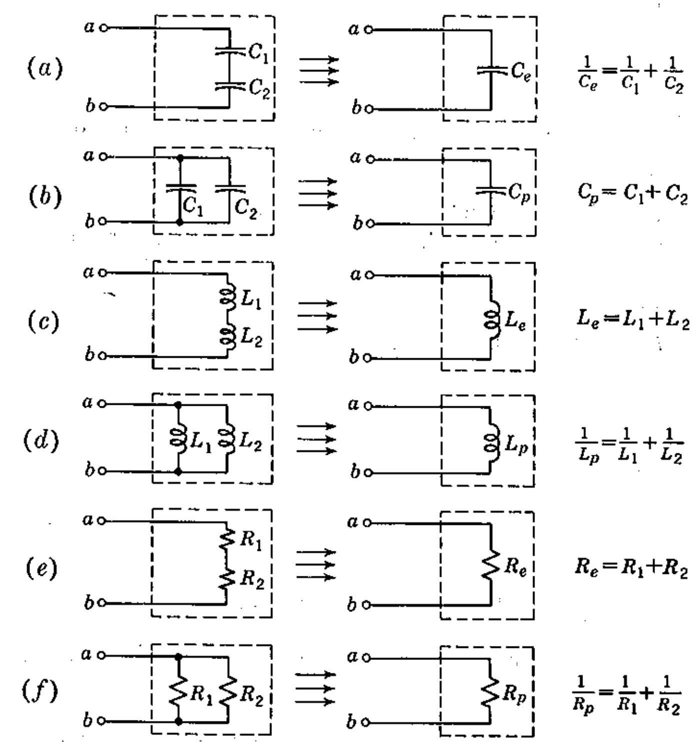

***Fig. 3-11* Equivalent elements.**

the relationship at the source terminals is

$$i_0 = 2.7 \frac{dv_o}{dt}$$

Hence it appears that the effect of the entire ladder at the source terminals is the same as if a 2.7-farad capacitance were connected in place of the ladder.

Series and parallel connection of similar elements It is left as an exercise for the reader to show that the equivalences shown in Fig. 3-11 are valid provided that, in those given by Fig. 3-11*a* and *d*, the proper initial conditions exist.

3-8 Voltage division and current division for similar elements

Consider the series connection of the two inductances shown in Fig. 3-12. (Although we are formally dealing with only two elements in series, the

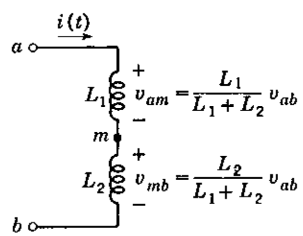

Fig. 3-12 Voltage division for series inductances.

reader will certainly recognize that each of these two elements may be the equivalent element for a combination of elements.) We desire to formulate the relationship between the voltage across one inductance and the voltage across both inductances.

The voltage across L_1 is, at any instant of time,

$$v_{am} = L_1 \frac{di}{dt}$$

and the voltage across L_2 is

$$v_{mb} = L_2 \frac{di}{dt}$$

Hence the ratio v_{am}/v_{mb} is equal to the ratio of the inductances,

$$\frac{v_{am}}{v_{mb}} = \frac{L_1}{L_2} \tag{3-22}$$

In words, when two inductances are connected in series, the voltage across each inductance is proportional to the value of that inductance.

Since the equivalent inductance of the two elements in series is

$$L_e = L_1 + L_2$$

we may also write

$$\frac{v_{am}}{v_{ab}} = \frac{L_1}{L_1 + L_2} \tag{3-23}$$

Equation (3-23) is the *voltage-division formula* for inductances. In words, this formula states that the ratio of the voltage across one inductance in a series connection of inductances is to the voltage across all the series-connected inductances as the one inductance is to the sum of the inductances.

For the series connection of several resistances, the same statement applies if the word inductance is replaced by resistance everywhere. This result is illustrated in Fig. 3-13. The proof is left as an exercise for the reader.

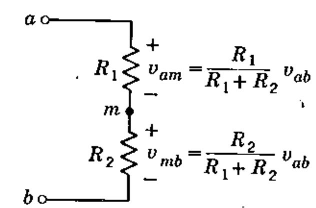

Fig. 3-13 Voltage division across series resistances.

In the case of capacitances, we can show that, if the same current has been flowing through the series-connected capacitances for all time, the voltages will divide as the reciprocals of the capacitances, i.e., proportionally to the elastances, as illustrated in Fig. 3-14. The proof is left an an exercise for the reader (Prob. 3-15).

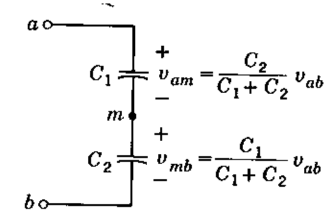

Fig. 3-14 Voltage division across series capacitances.

Consider now the parallel inductances L_1 and L_2 shown in Fig. 3-15. The current in each inductance is given, for all $t > 0$, by

$$i_1(t) = i_1(0) + \frac{1}{L_1} \int_0^t v_{ab} \, d\tau$$

and

$$i_2(t) = i_2(0) + \frac{1}{L_2} \int_0^t v_{ab} \, d\tau$$

If we now assume that the ratio $i_1(0)/i_2(0) = L_2/L_1$ [or that both $i_1(0)$ and $i_2(0)$ are zero], then

$$\frac{i_1(t)}{i_2(t)} = \frac{L_2}{L_1} \tag{3-24}$$

so that (under the assumed conditions) the current in two parallel inductances divides proportionally to the reciprocals of the inductances.

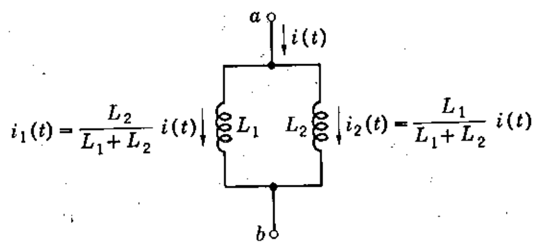

Fig. 3-15 Current division in parallel inductances.

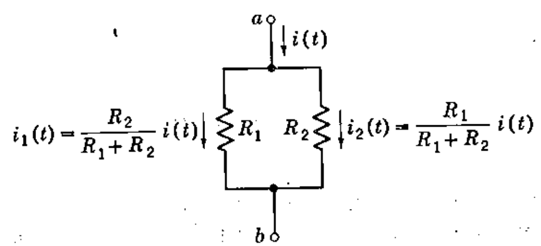

Fig. 3-16 Current division in parallel resistances.

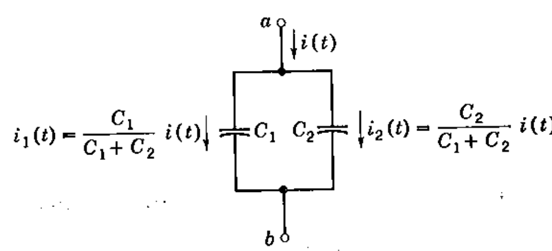

Fig. 3-17 Current division in parallel capacitances.

In the case of two parallel resistances (as shown in Fig. 3-16) the current divides proportionally to the conductances (reciprocal resistances), and for parallel capacitances the current divides proportionally to the capacitances as shown in Fig. 3-17. The proof of these statements is left as an exercise for the reader.

3-9 Equilibrium equations

We recall that, in general, any voltage or current which is not generated by an ideal source can be a network response; thus many responses are associated with a network. In most practical examples one is not interested in *every* response, but only in certain responses, which may constitute the "output." Thus the output of a network may be a voltage across several elements rather than the voltage across a single element. In such a case the desired response may not occur as a variable in the Kirchhoff-law equations (for example, it may be the voltage v_{ad} in Fig. 3-18, as discussed below,

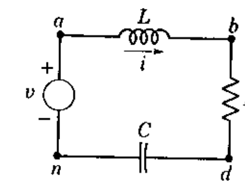

Fig. 3-18 A series R-L-C circuit excited by an ideal voltage source.

in Example 3-6). In other cases it may happen that the application of Kirchhoff's laws leads to simultaneous equations for several variables, only one of which is the desired response variable (Example 3-7). In either event one can manipulate the Kirchhoff-law equations to *derive an equation which relates the desired response function to the source function.* Such a derived equation is termed an *equilibrium equation;* the remainder of this chapter deals with efficient formulation of such an equation for series-parallel circuits; additional general techniques are given in Chaps. 11, 12, and 13.

Example 3-6 In the circuit of Fig. 3-18 obtain the equilibrium equation relating the response v_{ad} to the source v.

Solution From Eq. (3-4),

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{t} i\,d\tau = v \tag{3-25}$$

and from Fig. 3-18,

$$v_{ad} = L\frac{di}{dt} + Ri \tag{3-26}$$

Equations (3-25) and (3-26) are simultaneous equations for the variables v_{ad} and i. To eliminate i from the equations, substitute (3-26) into (3-25).

$$v_{ad} + \frac{1}{C}\int_{-\infty}^{t} i\,d\tau = v \tag{3-27}$$

Now differentiate each term in Eq. (3-27) with respect to t and solve for i:

$$i = C\frac{dv}{dt} - C\frac{dv_{ad}}{dt} \tag{3-28}$$

Substituting (3-28) in (3-25) yields

$$LC\frac{d^2v}{dt^2} - LC\frac{d^2v_{ad}}{dt^2} + RC\frac{dv}{dt} - RC\frac{dv_{ad}}{dt} + v - v_{ad} = v$$

or

$$LC\frac{d^2v_{ad}}{dt^2} + RC\frac{dv_{ad}}{dt} + v_{ad} = LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} \qquad (3\text{-}29)$$

Equation (3-29) is the desired equilibrium equation. We observe that it is not a Kirchhoff-law equation since sums of the individual terms do not represent voltages added up around a loop or currents entering or leaving a junction.

Example 3-7 In the circuit of Fig. 3-19 apply Kirchhoff's current law to obtain simultaneous equations for v_{an} and v_{bn} and obtain the equilibrium equation for v_{bn}.

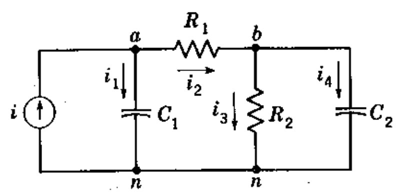

Fig. 3-19 Circuit for Example 3-7.

Solution Application of Kirchhoff's current law at junction a yields $i_1 + i_2 = i$, or

$$C_1\frac{dv_{an}}{dt} + \frac{1}{R_1}(v_{an} - v_{bn}) = i \qquad (3\text{-}30)$$

At junction b the result is $i_2 = i_3 + i_4$, or

$$\frac{1}{R_1}(v_{an} - v_{bn}) = \frac{1}{R_2}v_{bn} + C_2\frac{dv_{bn}}{dt} \qquad (3\text{-}31)$$

Equations (3-30) and (3-31) are the simultaneous differential equations for v_{an} and v_{bn}. From Eq. (3-31),

$$v_{an} = v_{bn} + \frac{R_1}{R_2}v_{bn} + R_1C_2\frac{dv_{bn}}{dt} \qquad (3\text{-}32)$$

Substituting for v_{an} from Eq. (3-32) into (3-30) gives

$$C_1\left(1 + \frac{R_1}{R_2}\right)\frac{dv_{bn}}{dt} + C_1R_1C_2\frac{d^2v_{bn}}{dt^2} + \frac{1}{R_2}v_{bn} + C_2\frac{dv_{bn}}{dt} = i$$

or collecting terms,

$$R_1C_1C_2\frac{d^2v_{bn}}{dt^2} + \frac{(R_1 + R_2)C_1 + R_2C_2}{R_2}\frac{dv_{bn}}{dt} + \frac{1}{R_2}v_{bn} = i \qquad (3\text{-}33)$$

Equation (3-33) is the required equilibrium equation.

At this point in the discussion we emphasize that the preceding examples are meant only to illustrate the idea of equilibrium equation. Efficient techniques for formulating such equations are discussed in the remainder of this chapter.

3-10 Operational notation

When a circuit consists of several types of passive elements in addition to sources, the formulation of equilibrium relations is no longer algebraic, but involves the manipulation of *simultaneous integrodifferential equations.* To facilitate such manipulations we use the *operational notation* of the calculus. The use of this notation allows the *manipulation* of integrodifferential equations in algebraic form.

We shall denote the operation "differentiate with respect to time" by the symbol p thus:

$$p \equiv \frac{d}{dt} \qquad \text{so that} \qquad p(f) \equiv \frac{df}{dt} \qquad (3\text{-}34)$$

The symbol p is termed differential operator. Using this operator, the second derivative is written as $p(pf) = d^2f/dt^2$. We write pp as p^2, exactly as if p were an algebraic quantity, thus:

$$p^n \equiv \frac{d^n}{dt^n} \qquad (3\text{-}35)$$

It can be shown[1] that simultaneous differential equations can be manipulated in the same way as simultaneous algebraic equations by writing them in operational form and then treating p as if it were an algebraic entity.

In the same manner that the symbol p is defined to mean differentiation with respect to time, we define division with p to mean integration with respect to time. We further define the operation p followed by $1/p$ *or* $1/p$ followed by p, operating on a function f, to mean

$$p \cdot \frac{1}{p}f = \frac{1}{p} \cdot pf = f \qquad (3\text{-}36)$$

Since integration involves an integration constant, Eq. (3-36) does not apply unless this constant is set to zero. In general, for inductance and capacitance, respectively,

$$i_{ab} = i_{ab}(0) + \frac{1}{L_{ab}}\int_0^t v_{ab}\,d\tau \qquad \text{or} \qquad v_{ab} = v_{ab}(0) + \frac{1}{C}\int_0^t i_{ab}\,d\tau \qquad (3\text{-}37)$$

we define

$$\frac{1}{p}(\ \ ) = \int_0^t (\ \ )\,d\tau \qquad (3\text{-}38)$$

so that for inductance L_{ab},

$$i_{ab} = i_{ab}(0) + \frac{1}{pL_{ab}}v_{ab} \qquad (3\text{-}39)$$

and for capacitance C_{ab},

$$v_{ab} = v_{ab}(0) + \frac{1}{pC_{ab}}i_{ab} \qquad (3\text{-}40)$$

It is shown in Chap. 6 that the initial values $i_{ab}(0)$ or $v_{ab}(0)$ can be represented by ideal sources and that, as a consequence, the formulation of *input-output* (source-response) equations is correctly carried out if the initial-condition terms in Eqs. (3-39) and (3-40) are set to zero. This statement also means that in the formulation of equilibrium equations (and net-

[1] See Chap. 8.

work functions; see Sec. 3-11) we *define a passive network as one consisting of R, L, and C elements, where the L's and C's store zero energy at $t = 0$*, so that the definition of $1/p$ given by Eq. (3-38) applies. In operational form, the voltage-current relationships for the elements R, L, and C are therefore

$$v = Ri \qquad v = pLi \text{ or } i = \frac{1}{pL}v \qquad i = pCv \text{ or } v = \frac{1}{pC}i \tag{3-41}$$

The *form* of these relationships is algebraic; the operator[1] pL plays the same role for inductance that R plays for resistance and $1/pC$ for capacitance. We now illustrate the derivation of equilibrium equations using operational notation.

We shall derive the equilibrium equations for $v_{ad}(t)$ in the resistive circuit of Fig. 3-20*a* and in the *R-L-C* circuit of Fig. 3-20*b* so as to show the similarity of technique when operational notation is used. We observe that each of the circuits consists of three passive elements in series with an ideal voltage source. To show the similarity of technique, we shall work out the solutions for both circuits side by side.

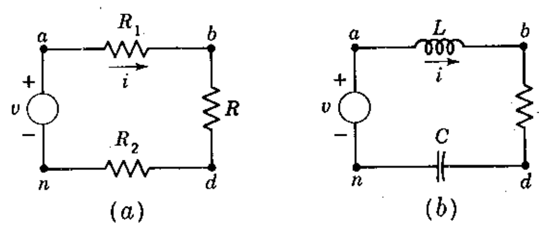

***Fig. 3-20** (a) A resistive circuit. (b) An R-L-C circuit with the same number of elements as the circuit of (a).*

For both circuits,

$$v_{ab} + v_{bd} + v_{dn} = v \tag{3-42}$$

For the circuit of Fig. 3-20a:

$$v_{ab} = R_1 i$$
$$v_{bd} = Ri$$
$$v_{dn} = R_2 i$$

Hence

$$(R_1 + R + R_2)i = v$$
$$i = \frac{1}{R_1 + R + R_2}v$$
$$v_{ad} = (R_1 + R)i$$
$$v_{ad} = \frac{R_1 + R}{R_1 + R + R_2}v \tag{3-43a}$$

For the circuit of Fig. 3-20b:

$$v_{ab} = pLi$$
$$v_{bd} = Ri$$
$$v_{dn} = \frac{1}{pC}i$$

Hence

$$\left(pL + R + \frac{1}{pC}\right)i = v$$
$$i = \frac{1}{pL + R + 1/pC}v$$
$$v_{ad} = (pL + R)i$$
$$v_{ad} = \frac{pL + R}{pL + R + 1/pC}v \tag{3-43b}$$

[1] For time-invariant elements the operator pL is identical with the operator Lp; that is, $pLi = d(Li)/dt = L\,di/dt = Lpi$. This is not true if L is a function of time.

Now, in the case of the resistive circuit, we can arrive at the result Eq. (3-43*a*) by voltage division:

$$\frac{v_{ad}}{v_{an}} = \frac{R_{ad}}{R_{an}} = \frac{R_1 + R}{R_1 + R + R_2} \tag{3-44}$$

The result for the analogous *R-L-C* circuit [Eq. (3-43*b*)] is seen to be related to the voltage-division method for resistive circuits as follows: Replace R_1 by pL and replace R_2 by $1/pC$ in Eq. (3-44). This gives

$$\frac{v_{ad}}{v} = \frac{pL + R}{pL + R + 1/pC} \tag{3-45}$$

Equation (3-45) is seen to be identical with Eq. (3-43*b*).

It is necessary now to emphasize the *difference* between Eq. (3-44) and Eq. (3-45). Equation (3-44) represents the *solution* of the problem: Find the response v_{ad} in terms of the source v. In the resistive circuit the *solution* is that v_{ad} is the fraction $(R_1 + R)/(R_1 + R + R_2)$ of v. On the other hand, Eq. (3-45) does not give v_{ad} numerically in terms of v, but rather, we have the equation

$$v_{ad} = \frac{pL + R}{pL + R + 1/pC}v \tag{3-46}$$

which, upon clearing the fraction, is seen to be

$$\left(pL + R + \frac{1}{pC}\right)v_{ad} = (pL + R)v$$

or, replacing p by d/dt and $1/p$ by the integral operation, we obtain the integrodifferential equation

$$L\frac{dv_{ad}}{dt} + Rv_{ad} + \frac{1}{C}\int_0^t v_{ad}\,d\tau = L\frac{dv}{dt} + Rv \tag{3-47a}$$

Differentiating both sides of this equation, we have

$$L\frac{d^2v_{ad}}{dt^2} + R\frac{dv_{ad}}{dt} + \frac{1}{C}v_{ad} = L\frac{d^2v}{dt^2} + R\frac{dv}{dt} \tag{3-47b}$$

which is identical with Eq. (3-29) as derived without the use of operational notation. The differentiation involved in going from Eq. (3-47*a*) to Eq. (3-47*b*) is identical with the "multiplication" with p of the numerator and denominator of the fraction in Eq. (3-46). Such multiplication (by Cp) results in

$$v_{ad}(t) = \frac{RCp + LCp^2}{RCp + LCp^2 + 1}v(t) \tag{3-48a}$$

This equation is interpreted as stating

$$(RCp + LCp^2 + 1)v_{ad}(t) = (RCp + LCp^2)v(t) \tag{3-48b}$$

which again is identical with Eq. (3-47*b*).

In summary, we conclude that the operational voltage-current relationships for inductance and capacitance, $v = pLi$ and $i = pCv$, respectively, allow the manipulation of integrodifferential relationships in algebraic form. *It is important to remember that the manipulations of integrodifferential relationships in algebraic form result in a (differential) equilibrium equation which must be solved to obtain the response in terms of the source.* In contrast, the analogous manipulations for the resistive network [as in the derivation of Eq. (3-43a)] result in the numerical relationship between response and source.

3-11 Introduction to operational network function

In Eq. (3-43a) the response v_{ad} of the resistive network of Fig. 3-20a is related to the source v by the formula

$$v_{ad} = \frac{R_1 + R}{R_1 + R + R_2} v$$

so that response v_{ad} is related to the source by the fraction $(R_1 + R)/(R_1 + R + R_2)$. In Eq. (3-48$a$) the analogous fraction $(LCp^2 + RCp)/(LCp^2 + RCp + 1)$ relates the response v_{ad} of the R-L-C circuit of Fig. 3-20b to the source $v(t)$. As described above, this fraction is a *differential operator* since Eq. (3-47) *must always* be interpreted as the differential equation (3-47b) [or its integrodifferential version (3-47a)]. The fraction $(LCp^2 + RCp)/(LCp^2 + RCp + 1)$ may be considered a function of the (operator) p. *When such a function of the operator p relates a response function in a network to a source function, the function of p is termed operational network function.* Denoting, in general, an operational network function by $H(p)$, we can write formally

$$\text{Response function} = [H(p)]\ \text{source function}$$

In the example of Fig. 3-20b the operational network function $H(p)$ which relates the response v_{ad} to the source $v(t)$ is given by

$$H(p) = \frac{LCp^2 + RCp}{LCp^2 + RCp + 1} \qquad v_{ad} = [H(p)]v$$

We can regard the resistive network as a special case where H is independent of p and is an algebraic, rather than a differential, operator.

Example 3-8 In the circuit of Fig. 3-21, find the network functions which relate i_4, v_{bn}, and v_{ab} to the source current i_s.

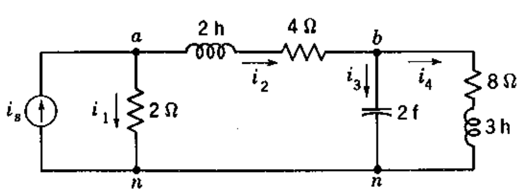

Fig. 3-21 ***Ladder network analyzed in Example 3-8.***

Solution As in resistive ladder networks, we begin with the relationship between i_4 and v_{bn}:

$$(8 + 3p)i_4 = v_{bn} \qquad i_4 = \frac{1}{8 + 3p} v_{bn} \tag{3-49}$$

$$i_3 = 2pv_{bn}$$

Hence, since $i_2 = i_3 + i_4$,

$$i_2 = \left(2p + \frac{1}{8 + 3p}\right) v_{bn} = \frac{6p^2 + 16p + 1}{8 + 3p} v_{bn}$$

Further,

$$v_{ab} = (2p + 4)i_2 = \frac{(2p + 4)(6p^2 + 16p + 1)}{8 + 3p} v_{bn} = \frac{12p^3 + 56p^2 + 66p + 4}{8 + 3p} v_{bn} \tag{3-50}$$

Since

$$v_{an} = v_{ab} + v_{bn}$$

we have

$$v_{an} = \left(\frac{12p^3 + 56p^2 + 66p + 4}{8 + 3p} + 1\right) v_{bn} = \frac{12p^3 + 56p^2 + 69p + 12}{8 + 3p} v_{bn}$$

Finally,

$$i_1 = \frac{v_{an}}{2} = \frac{12p^3 + 56p^2 + 69p + 12}{16 + 6p} v_{bn}$$

so that

$$i_s = i_2 + i_1 = \left(\frac{6p^2 + 16p + 1}{8 + 3p} + \frac{12p^3 + 56p^2 + 69p + 12}{16 + 6p}\right) v_{bn} = \frac{12p^3 + 68p^2 + 101p + 14}{16 + 6p} v_{bn}$$

or

$$v_{bn} = \frac{16 + 6p}{14 + 101p + 68p^2 + 12p^3} i_s$$

From Eq. (3-49),

$$i_4 = \frac{2}{14 + 101p + 68p^2 + 12p^3} i_s$$

and from Eq. (3-50),

$$v_{ab} = \frac{8 + 132p + 112p^2 + 24p^3}{14 + 101p + 68p^2 + 12p^3} i_s$$

so that the required network functions are H_i, H_a, and H_b, where

$$i_4 = [H_i(p)]i_s \qquad H_i(p) = \frac{2}{14 + 101p + 68p^2 + 12p^3}$$

$$v_{ab} = [H_a(p)]i_s \qquad H_a(p) = \frac{8 + 132p + 112p^2 + 24p^3}{14 + 101p + 68p^2 + 12p^3}$$

and

$$v_{bn} = [H_b(p)]i_s \qquad H_b(p) = \frac{16 + 6p}{14 + 101p + 68p^2 + 12p^3}$$

We recall again that the above results correspond to the differential equations

$$12\frac{d^3i_4}{dt^3} + 68\frac{d^2i_4}{dt^2} + 101\frac{di_4}{dt} + 14i_4 = 2i_s$$

$$12\frac{d^3v_{ab}}{dt^3} + 68\frac{d^2v_{ab}}{dt^2} + 101\frac{dv_{ab}}{dt} + 14v_{ab} = 8i_s + 132\frac{di_s}{dt} + 112\frac{d^2i_s}{dt^2} + 24\frac{d^3i_s}{dt^3}$$

$$12\frac{d^3v_{bn}}{dt^3} + 68\frac{d^2v_{bn}}{dt^2} + 101\frac{dv_{bn}}{dt} + 14v_{bn} = 16i_s + 6\frac{di_s}{dt}$$

3-12 *Driving point and transfer functions: immittance and gain*

The operational voltage-current relations at the terminals of a terminal-pair network are often of special interest. A network function which relates a *voltage* source to a *current* response, or vice-versa, is termed *operational immittance.*[1] When the voltage and current in question are those at the terminals of a terminal-pair network, the term operational *driving-point* immittance is used. When the source and response are *not* associated with the *same* pair of terminals, the network function is termed a *transfer function.* Thus, in Fig. 3-22*a*, the network function which relates the source at *m-b* to

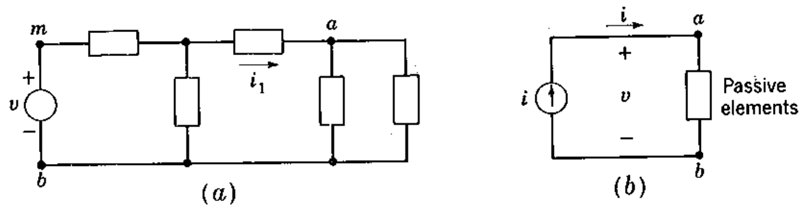

Fig. 3-22 (a) The network functions that relate v_{ab} or i_1 to v_{mb} is a transfer function. (b) Driving-point immittance relates v and i.

the response at *a-b* is a transfer function. This is in contrast to the situation described by Fig. 3-22*b*, where the source and response are associated with the same terminal pair *a-b*, and the combination of passive, initially unenergized elements which form the terminal pair *a-b* can be described by a driving-point immittance. If, in Fig. 3-22*b*, the source is a current source, the response v is written, symbolically,

$$v = [Z(p)]i \qquad (3\text{-}51)$$

where $Z(p)$ is termed *operational driving-point impedance.* When $v(t)$ is the source, the response is $i(t)$, and one writes

$$i(t) = [Y(p)]v \qquad Y(p) = \frac{1}{Z(p)} \qquad (3\text{-}51a)$$

where $Y(p)$ *is termed operational driving-point admittance.*

[1] Immittance is a synthetic word meaning *either* impedance or admittance.

For the three basic elements R, L, and C, we have

$$v = Ri \qquad v = pLi \qquad v = \frac{1}{pC}i$$

Hence

$$\begin{aligned} &\text{For a resistance:} && Z = R && Y = G = \frac{1}{R} \\ &\text{For an inductance:} && Z = pL && Y = \frac{1}{pL} \\ &\text{For a capacitance:} && Z = \frac{1}{pC} && Y = pC \end{aligned} \qquad (3\text{-}52)$$

Since Eq. (3-51) is analogous to $v = Ri$ and Eq. (3-51*a*) is analogous to

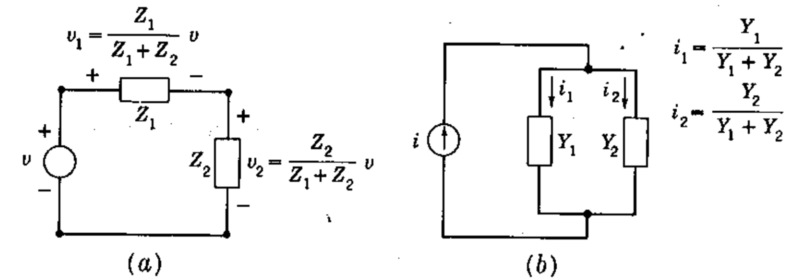

Fig. 3-23 (a) Voltage division for operational impedances. (b) Current division for operational admittances.

$i = Gv$, we conclude that operational impedances in series add and admittances in parallel add. Similarly, we can derive the operational voltage and current-division formulas by analogy: In the series circuit of Fig. 3-23*a*,

$$v_1 = \frac{Z_1(p)}{Z_1(p) + Z_2(p)}v(t) \qquad (3\text{-}53)$$

and in Fig. 3-23*b*,

$$i_1(t) = \frac{Y_1(p)}{Y_1(p) + Y_2(p)}i \qquad (3\text{-}54)$$

GAIN FUNCTIONS A transfer function may be an impedance or admittance function. For example, in Fig. 3-22*a*, the network function which relates the response $i_1(t)$ to the source $v(t)$ is a transfer admittance. If a transfer function relates a voltage (current) response to a voltage (current) source, it is referred to as a voltage (current) gain function. The reader may easily show that, in Fig. 3-24, $i_1(t) = \frac{1}{4}i(t)$. In this expression the network func-

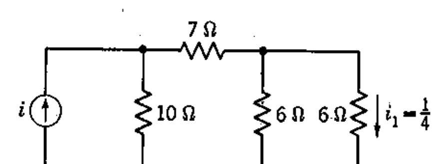

Fig. 3-24 Current gain in a resistive network.

tion $\frac{1}{a}$ relates the current response $i_1(t)$ to the source $i(t)$ and is therefore a current gain[1] function. Similarly, in Eq. (3-53), the transfer function $Z_1(p)/[Z_1(p) + Z_2(p)]$ is a voltage gain function relating the voltage response $v_1(t)$ to the voltage source $v(t)$.

We note that, in general, the reciprocal of a driving-point immittance is also a driving-point immittance, but the reciprocal of a transfer function has no significance as a network function.

Example 3-9 In the circuit of Fig. 3-25 derive the network function relating v_1 to v by operational voltage division.

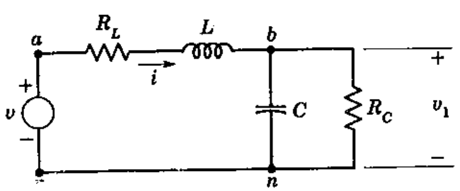

Fig. 3-25 Circuit for Example 3-9.

Solution From voltage division

$$v_1 = \frac{Z_{bn}(p)}{Z_{an}(p)} v$$

we formulate Z_{bn} and Z_{an}. The admittance Y_{bn} is given by

$$Y_{bn}(p) = \frac{1}{R_C} + pC = \frac{1 + pCR_C}{R_C}$$

Since the driving-point impedance Z_{bn} is the reciprocal of the driving-point admittance Y_{bn}, we have

$$Z_{bn}(p) = \frac{R_C}{1 + pR_CC}$$

Since

$$Z_{ab}(p) = R_L + pL$$

$$Z_{an}(p) = Z_{ab}(p) + Z_{bn}(p) = R_L + pL + \frac{R_C}{1 + pR_CC}$$

$$= \frac{p^2R_CLC + p(R_LR_CC + L) + R_L + R_C}{1 + pR_CC}$$

Hence

$$v_1 = \frac{R_C}{p^2R_CLC + p(R_LR_CC + L) + R_L + R_C} v = H(p)v$$

which is the desired result. Note that in the above expression $H(p)$ is a transfer (voltage-gain) function, and in contrast with the relationship $Z_{bn} = 1/Y_{bn}$ used above, the reciprocal of transfer function $H(p)$ is not a transfer function of the network.

[1] Loss function would be a more appropriate term in this case. In general, however, gain functions are differential operators, and not numbers, and distinction between gain and loss is not relevant.

3-13 Introduction to the solution of equilibrium equations

The foregoing discussion and examples illustrate procedures for the formulation of network functions or the corresponding equilibrium equations. We shall now discuss some of the general properties of such functions and equations.

FORM OF EQUILIBRIUM EQUATION The equilibrium equation which relates a response function to source functions in a linear network shows how *a certain linear combination of response function with its derivatives is related to another linear combination of the source function(s) with its (their) derivatives.* Thus, for example, in the circuit of Fig. 3-20*b*, from Eq. (3-47*b*), we have that the linear combination of the response v_{ad} with its derivatives,

$$\left(Lp^2 + Rp + \frac{1}{C}\right) v_{ad}$$

is equal to

$$(Lp^2 + Rp)v$$

which is seen to be a linear combination of the source function and its derivatives. In Chap. 11 it is shown that this general form always arises; without going through details of the proof, we may anticipate this result from the fact that all voltages and currents in the network are linearly related by algebraic or differential operations. The reader can verify the form for all the preceding examples. Before proceeding, however, we present an example involving more than one source function.

Example 3-10 In the circuit of Fig. 3-26 establish the equilibrium equation relating the response $v_2(t)$ to the source functions $i(t)$, $v_a(t)$, and $v_b(t)$.

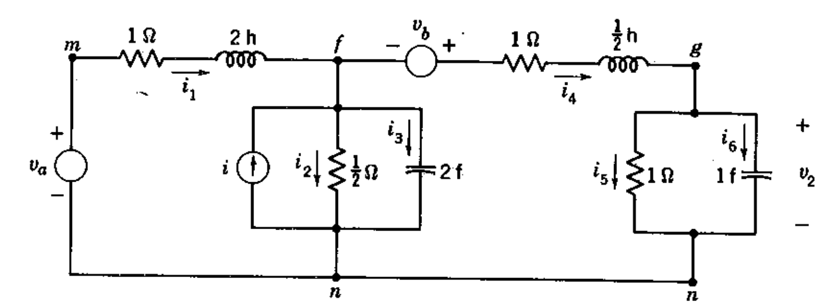

Fig. 3-26 Circuit with three source functions used in Example 3-10.

Solution Since $i_5 = v_2/1$ and $i_6 = pv_2$, we have, from the current law,

$$i_4 = (1 + p)v_2$$

and

$$v_{f_g} = -v_b + (1 + \tfrac{1}{2}p)(1 + p)v_2$$
$$= -v_b + (\tfrac{1}{2}p^2 + \tfrac{3}{2}p + 1)v_2$$

But

$$v_{fn} = v_{fg} + v_2$$
$$= -v_b + (\tfrac{1}{2}p^2 + \tfrac{3}{2}p + 2)v_2$$

and

$$i_2 = 2v_{fn} \qquad i_3 = 2pv_{fn} \qquad i_2 + i_3 = (2 + 2p)v_{fn}$$

and also

$$\begin{aligned} i_1 &= -i + i_2 + i_3 + i_4 \\ &= -i + (2 + 2p)[-v_b + (\tfrac{1}{2}p^2 + \tfrac{3}{2}p + 2)v_2] + (1 + p)v_2 \\ &= -i - (2 + 2p)v_b + (p^3 + 4p^2 + 8p + 5)v_2 \\ v_{mf} &= (1 + 2p)i_1 \\ &= -(1 + 2p)i - (4p^2 + 6p + 2)v_b + (2p^4 + 9p^3 + 20p^2 + 18p + 5)v_2 \end{aligned}$$

so that

$$\begin{aligned} v_{mn} &= v_{mf} + v_{fn} \\ &= -(1 + 2p)i - (4p^2 + 6p + 3)v_b + (2p^4 + 9p^3 + \tfrac{41}{2}p^2 + \tfrac{39}{2}p + 7)v_2 \end{aligned}$$

Since $v_{mn} = v_a$, we have, finally,

$$(2p^4 + 9p^3 + \tfrac{41}{2}p^2 + \tfrac{39}{2}p + 7)v_2 = v_a + (1 + 2p)i + (4p^2 + 6p + 3)v_b \qquad (3\text{-}55)$$

We observe that the left side of Eq. (3-55) is a linear combination of v_2 with its derivatives and the right side is a linear combination of the source functions with their derivatives.

FORM OF NETWORK FUNCTION A network function which relates a selected network response variable (such as v_2 of the foregoing example) to a single source function (such as v_a in the example) can be written as the ratio of two polynomials in p. Thus, in general,

$$H(p) = \frac{N(p)}{D(p)} \qquad (3\text{-}56)$$

where both $N(p)$ and $D(p)$ are polynomials in p, that is,

$$N(p) = b_0 + b_1p + \cdots + b_np^m$$

$D(p) = a_0 + a_1p + \cdots + a_np^n$, so that

$$H(p) = \frac{b_np^m + \cdots + b_1p + b_0}{a_np^n + \cdots + a_1p + a_0}$$

The above observation follows directly from the nature of the equilibrium equation. Denoting a response by $y(t)$ and a source by $x(t)$,

$D(p)y(t)$ = linear combination of response function y with its derivatives
$N(p)x(t)$ = linear combination of source function x with its derivatives

Since

$$D(p)y(t) = N(p)x(t)$$
$$y(t) = \frac{N(p)}{D(p)}x(t)$$

as stated above.

FORCING FUNCTION In the equation

$$[D(p)]y(t) = [N(p)]x(t)$$

the terms $[N(p)]x(t)$ describe how the source function x influences the response y. Thus if, as in Eq. (3-55), we assume $v_a \equiv 0$, $i \equiv 0$, $x = v_b$, and $y = v_2$, we have the equation

$$(2p^4 + 9p^3 + \tfrac{41}{2}p^2 + \tfrac{39}{2}p + 7)v_2 = (4p^2 + 6p + 3)v_b$$

where the terms $(4p^2 + 6p + 3)v_b$ describe the relative weights with which v_b and its first two derivatives influence v_2. This linear combination of the source function with its derivatives is termed a *forcing function* of the equilibrium equation. Thus, in general, the sum of terms in an equilibrium equation which represent linear combination of the source functions with their derivatives is termed the *forcing function* of the equilibrium equation. Denoting the forcing function by $f(t)$, we have, for example, in Eq. (3-55),

$$(2p^4 + 9p^3 + \tfrac{41}{2}p^2 + \tfrac{39}{3}p + 7)v_2 = f(t) \qquad (3\text{-}57a)$$

where, for the response $v_2(t)$ of the network of Fig. 3-26,

$$f(t) = v_a + (1 + 2p)i + (4p^2 + 6p + 3)v_b \qquad (3\text{-}57b)$$

We note that *if the source functions are all identically zero, the forcing function must be zero.* In passing, it is pointed out that the converse of the above is *not true.* If in Eq. (3-57b) $v_a \equiv 0$ and $v_b \equiv 0$ but $i \not\equiv 0$, we can still have $f(t) \equiv 0$ if $i(t) = Ae^{-t/2}$, where A is any constant. If $f(t) \equiv 0$ when all the source functions are not zero, a *resonance* is said to exist (Chap. 10).

Superposition If a forcing function $f(t)$ consists of, or can be represented as, a sum of forcing functions, the response to $f(t)$ can be represented as the sum of the responses to the components of f. That is, if

$$(a_0 + a_1p + \cdots + a_np^n)v_2(t) = f_a(t) + f_b(t) = f(t)$$

and if $(a_0 + a_1p + \cdots + a_np^n)v_{2a}(t) = f_a(t)$
and $(a_0 + a_1p + \cdots + a_np^n)v_{2b}(t) = f_b(t)$
then since $a_kp^kv_{2a} + a_kp^kv_{2b} = a_kp^k(v_{2a} + v_{2b})$

$$(a_0 + a_1p + \cdots + a_np^n)(v_{2a} + v_{2b}) = f(t)$$

and the response is the sum $v_2 = v_{2a} + v_{2b}$.

PARTICULAR AND COMPLEMENTARY SOLUTIONS The ensuing sections and chapters deal with the development of efficient techniques for the solution of equilibrium equations. The remaining discussion in this section is intended to be a general introduction to this subject.

Since in the equation

$$(a_0 + a_1p + a_2p^2 + \cdots + a_np^n)v_2(t) = f(t) \qquad (3\text{-}58)$$

one can *always* interpret $f(t)$ as the sum $f(t) + 0$, the solution of Eq. (3-58), if it is to be complete, must include the solution of the homogeneous equation

$$(a_0 + a_1 p + \cdots + a_n p^n)v_1(t) = 0$$

Since we associate zero forcing function with *source-free* networks, we term the solution of the homogeneous equation the *source-free component of the response* and often use the subscript f to denote this component. Thus, for

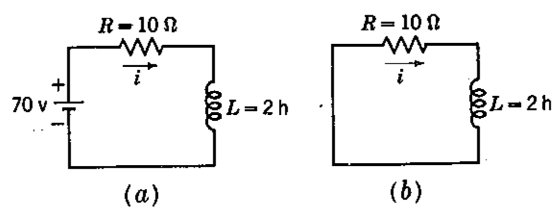

***Fig. 3-27** (a) An R-L circuit with a source. (b) Source-free circuit for (a).*

example, the equilibrium equation for the current of Fig. 3-27*a* is

$$2\frac{di}{dt} + 10i = 70 \tag{3-59}$$

In this equation $i(t)$ has as a component the solution $i_f = Ke^{-5t}$, where

$$2\frac{di_f}{dt} + 10i_f = 0$$

which is the solution to the source-free circuit of Fig. 3-27*b*. We note further that Eq. (3-59) has a solution $i_s = 7$, which is the current in the circuit of Fig. 3-27*a* when the time variations of $i_f = Ke^{-5t}$ have died out as t approaches infinity. Furthermore, the linear combination

$$i = 7 + Ke^{-5t}$$

satisfies Eq. (3-59). In the theory of differential equations, the solution $i = 7$ is referred to as a *particular* solution of Eq. (3-59). In circuit analysis, this term is referred to as *response due to the source.* We note that $i = 7 + 14e^{-5t}$ is also a particular solution, whereas $i = 7 + Ke^{-5t}$ with K an unspecified (arbitrary) constant represents the *totality of solutions.* Generalization of the concepts illustrated in the above example leads to the following mathematical definitions:

Particular solution: Any solution of the inhomogeneous equation.
Complementary solution: The general solution of the homogeneous equation.
General solution, or "totality of solutions": A linear combination of the particular and complementary solutions.

In network theory there are two particular solutions of special interest, the component of the response due to the source and the complete response.

The component of the response due to the source is that particular solution which, *independently of initial conditions,* vanishes when the forcing function vanishes. Thus, in the equation

$$2\frac{di}{dt} + 10i = 70$$

the solution $i = 7 = i_s$ is the component of the response due to the source. We shall often distinguish this part of the response by the subscript s, as in i_s or $[i(t)]_s$ or $[v(t)]_s$.

Complete response of a network We recall that, in the absence of an impulsive voltage source in series with an inductance, the current through the inductance must be a continuous function of time. Therefore, for the example of Fig. 3-27, the value of K in the solution $i(t) = 7 + Ke^{-5t}$ must be evaluated so that the current (in the inductance) is continuous. Thus, if we are given $i(0^-) = I_0$, it follows that

$$i(0^+) = i(0^-) = I_0 = 7 + Ke^{-0} = 7 + K$$
$$K = I_0 - 7$$

and

$$i(t) = 7 + (I_0 - 7)e^{-5t}$$

The above expression for $i(t)$ has the following properties:

1 It satisfies the equilibrium equation of the circuit given in Eq. (3-59).
2 It satisfies the specified initial condition $i(0^+) = I_0$.

A solution of an equilibrium equation which satisfies the above two conditions is referred to as a *complete response.*

3-14 Response due to constant sources

In the example of the circuit of Fig. 3-27*a* we noted that the equilibrium equation for $i(t)$ is

$$10i + 2\frac{di}{dt} = 70 \tag{3-59}$$

and the complete response for $i(0) = I_0$ is

$$i(t) = i_s + i_f = 7 + (I_0 - 7)e^{-5t} \tag{3-60}$$

In the above solution the term $i_s = 7$ is obtained by reasoning that since the source (70 volts) is constant, the response due to the source, i_s, will also be constant, and $di_s/dt \equiv 0$. Substituting zero for the term di/dt in Eq. (3-58*a*), we have $10i_s = 70$. We now observe that Eq. (3-59) may be written

$$i(t) = \frac{1}{10 + 2p}70 \tag{3-61}$$

Substitution of $di/dt \equiv 0$ (for constant source) in Eq. (3-59) is equivalent to setting p equal to zero in Eq. (3-61). This results in

$$i_s = \frac{1}{10 + 2p}\bigg|_{p=0} 70 = 7$$

We now generalize the result of this example by stating that, if the source $x = X_0$ in a network is constant (time-invariant) and a response $y(t)$ is given by

$$y(t) = [H(p)]x(t) = [H(p)]X_0$$

then the component of the response due to the constant source is given as

$$y_s = Y_s = H(0)X_0 \qquad \text{if } \frac{1}{H(0)} \neq 0$$

As an example for the circuit of Fig. 3-26, the response $v_2(t)$ due to sources v_a, v_b, and i is given by Eq. (3-55). Using network-function symbolism, this equation may be rewritten

$$v_2(t) = H_a(p)v_a + H_b(p)v_b + H_i(p)i \tag{3-62}$$

Designating $D(p)$ by

$$D(p) = 2p^4 + 9p^3 + 20.5p^2 + 19.5p + 7 \tag{3-63}$$

from Eq. (3-55) we have

$$v_2(t) = \frac{1}{D(p)}v_a + \frac{4p^2 + 6p + 3}{D(p)}v_b + \frac{1 + 2p}{D(p)}i \tag{3-64}$$

If the sources v_a, v_b, and i are constant, say, $v_a = V_a = 70$ volts, $v_b = V_b = 14$ volts, $i = I = 21$ amp, then, in accordance with the above discussion the component of response due to these *constant* sources is

$$v_{2s} = H_a(0)v_a + H_b(0)v_b + H(0)i$$

Replacing p with zero in expressions for $H_a(p)$, $H_b(p)$, and $H(p)$, we have

$$v_{2s} = \tfrac{1}{7} \times 70 + \tfrac{3}{7} \times 14 + \tfrac{1}{7} \times 21 = 19 \text{ volts}$$

The foregoing procedure for calculation of response due to constant sources, illustrated by this example, is based on the theory of differential equations where it is known that a particular solution of an inhomogeneous linear differential equation with constant coefficients can generally be constructed as a *linear combination of the forcing function with all its derivatives.* When the forcing function $f(t)$ is constant, F_0, the equation

$$(a_n p^n + a_{n-1}p^{n-1} + \cdots + a_1 p + a_0)v(t) = F_0$$

is seen to have as solution the constant

$$v(t) = V = \frac{F_0}{a_0}$$

provided that $a_0 \neq 0$. This statement is verified by direct substitution.

In a network with a constant source $v_1(t) = V_1$, if the forcing function is given as $N(p)v_1(t)$, we have

$$\begin{aligned} f(t) &= (b_0 + b_1 p + \cdots + b_m p^m)v_1(t) \\ &= b_0 V_1 = F_0 \end{aligned}$$

and the response due to the source v (denoted by the subscript s) is given by

$$v_{2s} = V_2 = \frac{b_0}{a_0}V_1$$

As mentioned above, if in

$$H(p) = \frac{b_0 + b_1 p + \cdots + b_m p^m}{a_0 + a_1 p + \cdots + a_n p^n}$$

we set $p = 0$, $H(0) = b_0/a_0$, and

$$v_{2s} = \frac{b_0}{a_0}V_1 = H(0)V_1 \qquad a_0 \neq 0$$

As mentioned before, replacement of p with zero, when the source is constant, is due to the fact that all derivatives of constants are zero. In the above discussion it is assumed that $a_0 \neq 0$. When $a_0 = 0$ and $a_1 \neq 0$, one may define $pv = y$ and solve for the constant $dv/dt = F_0/a_1$.

3-15 Response due to exponential sources: the special role of exponential functions in linear analysis

The exponential function is unique in that it is the only function whose derivative is proportional to itself. Thus, if

$$\begin{aligned} f(t) &= Ke^{st} \\ pf(t) &= sKe^{st} = sf(t) \end{aligned} \tag{3-65}$$

Equation (3-65), $pf = sf$, should not be read as $p = s$. Rather, it states that, when f is exponential, the operation of differentiation with respect to time is identical with multiplication by s.

Below we show that,[1] if the source in a network is exponential and of the form

$$x(t) = X_g e^{s_g t} \tag{3-66}$$

where the subscript g refers to "generator" and both V_g and s_g are constants, then if the response $y(t)$ is related to this source by $H(p)$,

$$y(t) = H(p)x(t) = H(p)X_g e^{s_g t}$$

it follows that the response due to the source is

$$y_s(t) = H(s_g)X_g e^{s_g t} \qquad \frac{1}{H(s_g)} \neq 0$$

For example, if in the circuit of Fig. 3-26 the source $v_b(t) = V_g e^{s_g t} = 20e^{-2t}$, $v_a(t) = 70$ (a constant source), and $i(t) \equiv 0$, then from

$$v_2(t) = H_a(p)v_a + H_b(p)v_b + H(p)i$$

and the corresponding expressions for H_a, H_b, and H given in Eq. (3-64), we have

$$\begin{aligned} v_2(t)_s &= H_a(0)v_a + H_b(s_g)v_b + 0 \\ &= \frac{1}{2p^4 + 9p^3 + 20.5p^2 + 19.5p + 7}\bigg|_{p=0} \times 70 \\ &\qquad + \frac{4p^2 + 69p + 3}{2p^4 + 9p^3 + 20.5p^2 + 19.5p + 7}\bigg|_{p=s_g=-2} \times 20e^{-2t} \\ &= \tfrac{1}{7} \times 70 + \tfrac{7}{10} \times 20e^{-2t} = 10 + 14e^{-2t} \end{aligned}$$

[1] With certain exceptional cases discussed in Chap. 8.

In this result we note that such a constant source $v_a = 70$ may be considered to be exponential with $s_g = 0$; that is,

$$70 = 70e^{(0)t}$$

The case of response due to a constant source is a special case of the response due to exponential sources.

To show that the procedure used above is general, we note that, when a source $v_1(t)$ has the form $v_1(t) = V_1e^{s_gt}$, the forcing function is given by

$$f(t) = [N(p)]v_1(t) = (b_0 + b_1p + b_2p^2 + \cdots + b_mp^m)v_1(t)$$

But since $pv_1 = s_gv_1$, for exponential v_1,

$$f(t) = (b_0 + b_1s_g + b_2s_g^2 + \cdots + b_ms_g^m)V_1e^{s_gt} = N(s_g)V_1e^{s_gt}$$

so that $$f(t) = F_1e^{s_gt} \qquad F_1 = N(s_g)V_1$$

Thus, for example, if $(2p^2 + 8p + 58)v_2 = (4p + 7)e^{-4t}$, then

$$f(t) = [4 \times (-4) + 7]e^{-4t} = -9e^{-4t}$$

If

$$D(p)v_2 = (a_0 + a_1p + a_2p^2 + \cdots + a_np^n)v_2 = F_1e^{s_gt} \qquad (3\text{-}67)$$

is to have a solution

$$v_2 = V_2e^{s_gt} \qquad (3\text{-}68)$$

then $$D(p)v_2 = (a_0 + a_1p + \cdots + a_np^n)V_2e^{s_gt} = F_1e^{s_gt}$$
but $$p^kV_2e^{s_gt} = s_g^kV_2e^{s_gt}$$

so that (3-68) is a solution of (3-67) if

$$D(s_g)V_2e^{s_gt} = (a_0 + a_1s_g + a_1s_g^2 + \cdots + a_ns_g^n)V_1e^{s_gt} = F_1e^{s_gt}$$
or if $$D(s_g)V_2 = F_1$$

which is possible wherever $D(s_g) \neq 0$ by letting

$$V_2 = \frac{F_1}{D(s_g)} \qquad D(s_g) \neq 0$$

We now recall from the relations above that if $v_1 = V_1e^{s_gt}$,

$$F_1 = N(s_g)V_1$$

so that $$V_2 = \frac{N(s_g)}{D(s_g)}V_1 = H(s_g)V_1$$

Thus, in general, if

$$v_2(t) = H(p)v_1(t)$$

and when $v_1(t)$ is the exponential source function,

$$v_1(t) = V_1e^{s_gt}$$

and if $1/[H(s_g)] \neq 0$, then[1] the response due to the source will be proportional to the source, and the proportionality constant will be the network

[1] This is the exceptional case mentioned above.

function with the operator p replaced by the number s_g:

$$v_{2s} = H(s_g)V_1e^{s_gt} \qquad D(s_g) \neq 0$$

[The special case $D(s_g) = 0$ is treated in Chap. 8.]

Thus, for example, if

$$(2p^2 + 8p + 58)v_2 = (4p + 7)e^{-4t}$$
$$v_{2s} = \frac{4p + 7}{2p^2 + 8p + 58}e^{-4t}$$

and the response component due to the source is

$$v_{2s} = \frac{(4)(-4) + 7}{2 \times 16 + 8(-4) + 58}e^{-4t} = -\tfrac{9}{58}e^{-4t}$$

PROBLEMS

3-1 A 10-ohm resistance is connected in series with a 2-henry inductance. Current in this combination is given by $i(t) = 2t$. Calculate (*a*) the magnitude of voltage across the combination at $t = 0$ and at $t = 1$; (*b*) the energy transferred to the series combination in the time interval $t = 0$ to $t = 1$.

3-2 In the series *R-L-C* circuit shown in Fig. 3-7, let $R = 2$ ohms, $L = 2$ henrys, and $C = \frac{1}{2}$ farad. Calculate and sketch $v_{an}(t)$ if (*a*) $i(t) = 2u(t)$; (*b*) $i(t) = 2tu(t)$; (*c*) $i(t) = e^{-2t}u(t)$. The capacitance is initially uncharged.

3-3 In the parallel *R-L-C* circuit shown in Fig. 3-8, let $R = 1$ ohm, $L = 1$ henry, and $C = 1$ farad. Calculate and sketch $i(t)$ if (*a*) $v(t) = u(t)$; (*b*) $v(t) = 3tu(t)$; (*c*) $v(t) = (2 + 3t)u(t)$; (*d*) $e^{-2t}u(t)$. The inductance carries zero current at $t = 0$.

3-4 (*a*) In the circuit of Fig. P3-4, apply Kirchhoff's current law at junction b, using v as the variable, and obtain the differential equation that relates v to v_o. *Hint:* Note that $i_{ab} = (v_o - v)/12$. (*b*) Sketch the waveform of v_o if v is given by (1) $v = u(t)$; (2) $v = tu(t)$; (3) $v = (1 + t)u(t)$; (4) $e^{-2t/3}$; (5) $e^{-2t/3}u(t)$.

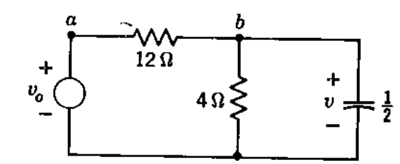

Fig. P3-4

3-5 In the circuit of Fig. P3-5: (*a*) Express i_2 in terms of i_1. (*b*) Express i in terms of i_1. (*c*) Express v_{ab} in terms of i_1. (*d*) Find the integrodifferential equation that relates i_1 to v. (*e*) Find the waveform of v if $i_1(t) = 2tu(t)$.

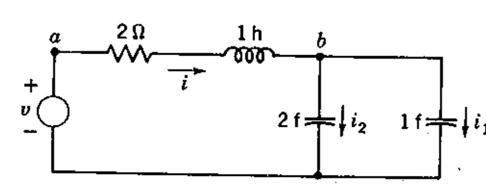

Fig. P3-5

3-6 (*a*) In the circuit of Fig. P3-6, apply Kirchhoff's voltage law on loop *a-b-n-a* and obtain the differential equation that relates i to i_s. *Hint:* Express the current in the 4-ohm resistance in terms of i and i_s. (*b*) Find the waveform of i_s if i is given by (1) $i = (2 + t^2)u(t)$; (2) $i = e^{-2t}$; (3) $i = e^{-2t}u(t)$; (4) $(e^{-2t} - 1)u(t)$.

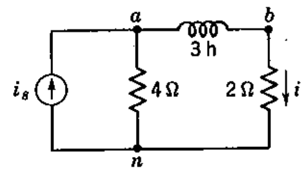

Fig. P3-6

3-7 (*a*) In the circuit of Fig. P3-7 show by use of Kirchhoff's current law at junction *a* that $i = 4v_1 + 3v_2 - 7v_{ab}$. (*b*) Show that $i = 2v_{ab} + 2dv_{ab}/dt$. (*c*) Use the results of parts *a* and *b* to find the differential equation that relates v_{ab} to the sources v_1 and v_2.

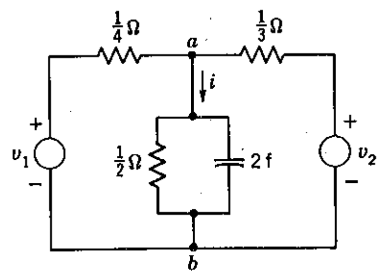

Fig. P3-7

3-8 In Fig. P3-6 replace the 4-ohm resistance by a 4-henry inductance and show that $7di/dt + 2i = 4di_s/dt$.

3-9 In the circuit of Fig. P3-9, let $R_1 = R_2 = 5$ ohms, $R_3 = 60$ ohms, $R_4 = R_5 = 10$ ohms, and replace R_6 and R_7 with open circuit ($R_{6,7} \to \infty$). Calculate (*a*) the ratio i_3/i_s; (*b*) the ratio v_{dn}/i_s; (*c*) the ratio v_{an}/i_s.

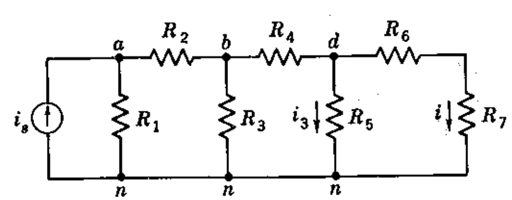

Fig. P3-9

3-10 In the circuit of Fig. P3-9 it is known that $i(t) = I = 2$ when $i_s = I_s = 15$. Find $i(t)$ if i_s is given by (*a*) $i_s = 3t$; (*b*) $i_s = 10 \cos 5t$.

3-11 In the circuit of Fig. P3-9 prove that the equivalent resistance of the seven elements at the source terminals is less than R_1 and greater than $R_1R_2/(R_1 + R_2)$.

3-12 In the capacitive ladder network shown in Fig. P3-12: (*a*) Calculate v/v_o. (*b*) Obtain the relationship between i and v_o.

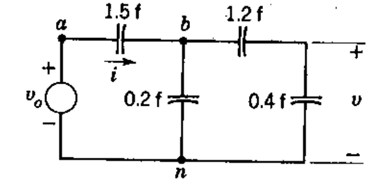

Fig. P3-12

3-13 In the inductive ladder circuit shown in Fig. P3-13, calculate (*a*) v_2/v_o; (*b*) v_{an}/v_o; (*c*) the relationship between v_o and i.

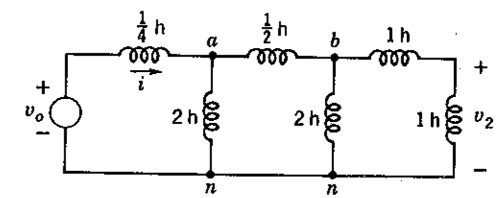

Fig. P3-13

3-14 In the resistive ladder circuit shown in Fig. P3-14, obtain a formula relating the sources v_A, v_B, and i_C to v_2.

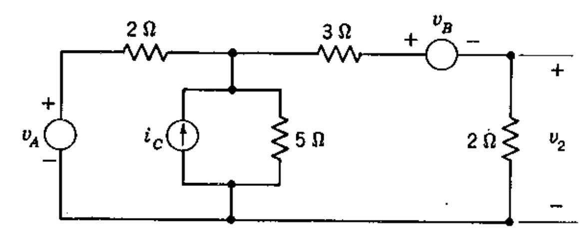

Fig. P3-14

3-15 (*a*) Prove the relationships indicated in Fig. 3-11. (*b*) Prove the voltage-division formula for series resistance and for series capacitance indicated in Figs. 3-13 and 3-14, respectively. (*c*) Prove the current-division formulas for parallel resistances and capacitances indicated in Figs. 3-16 and 3-17, respectively.

3-16 Calculate the equivalent inductance L_{ab} of Fig. P3-16 if $L = \frac{1}{4}$ henry, $L_1 = \frac{1}{2}$ henry, and L_2 is variable. Calculate also the largest and smallest value L_{ab} can have as L_2 is varied.

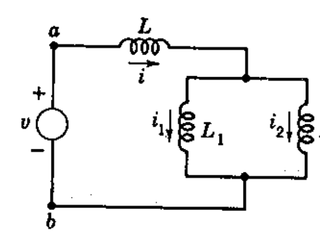

Fig. P3-16

3-17 In the circuit of Fig. P3-16, let $L = 0$ and set $i_1(0) = i_2(0) = 0$. The source waveform is the sawtooth pulse $v(t) = tu(t)u(T - t)$. It is specified that neither i_1 nor i_2 may exceed 8 amp. Calculate the largest allowable value of T and the maximum allowable value of i if $L_1 = 4L_2 = \frac{1}{2}$ henry.

3-18 The equivalent resistance of two resistances in series is 10 ohms. When the two resistances are placed in parallel, the equivalent resistance is 2.4 ohms. Calculate the values of the resistances.

3-19 The equivalent capacitance of two capacitances in series is 10 μf. When the same capacitances are connected in parallel, the equivalent capacitance is 50 μf. Calculate the values of the individual capacitances.

3-20 In the two circuits shown in Fig. P3-20 the same current I_0 flows for the same voltage V_{ab}. It is also known that the ratio I_1/I_0 is the same in both circuits. Calculate R_a and R_b.

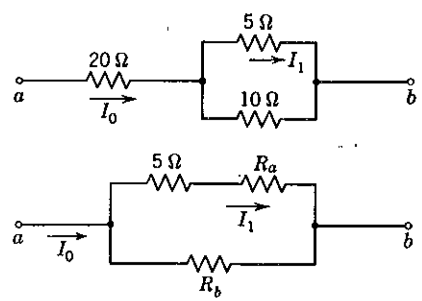

Fig. P3-20

3-21 In the circuit shown in Fig. P3-21, $v_{db} = 50$ mv when $v_{ab} = 150$ volts. Calculate R (voltmeter "multiplier").

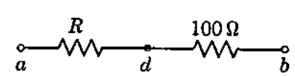

Fig. P3-21

3-22 In the circuit shown in Fig. P3-22, $I_1 = 100$ μa when $I = 5$ amp. Calculate R (ammeter "shunt").

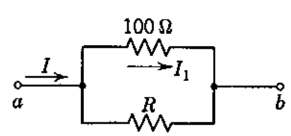

Fig. P3-22

3-23 (*a*) Two uncharged capacitances C_1 and C_2 are connected in series. If $C_1 = 2$ μf and C_2 is adjustable from 0.01 to 0.1 μf, calculate the maximum and the minimum values of the capacitance of the series combination. (*b*) Calculate the maximum and minimum values of the equivalent capacitance if the two capacitances are placed in parallel.

3-24 In the circuit shown in Fig. P3-24, the voltage value indicated below each capacitance value specifies the highest permissible voltage across the capacitance. Calculate the maximum permissible value of v_{ab}.

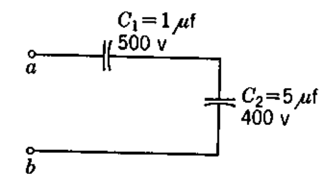

Fig. P3-24

3-25 In the circuit of Fig. P3-25, calculate V_{ab}.

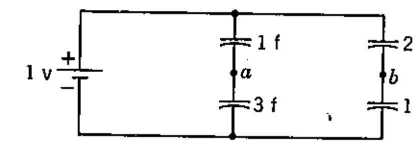

Fig. P3-25

3-26 In the circuit of Fig. P3-26, calculate I if (*a*) $R = 3$ ohms; (*b*) $R = 1.5$ ohms.

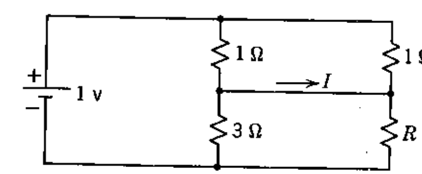

Fig. P3-26

3-27 In the circuit of Fig. P3-27, let $R = 0$ and calculate the ratio v_{ab}/v.

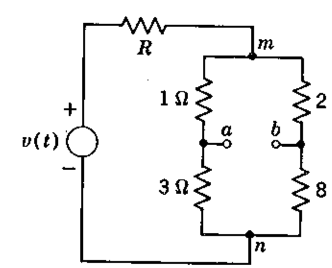

Fig. P3-27

3-28 In the circuit of Fig. P3-27, let $R = 1$ ohm and calculate the ratio v_{ab}/v.

3-29 In the circuit of Fig. P3-27, let $R = 1$ ohm and place a short circuit (zero resistance) between terminals *a-b*. Calculate the current in the short circuit I_{ab} if

3-30 In the circuit of Fig. P3-30, $v_{an}/v_o = \frac{1}{2}$ and $v_{bn}/v_o = \frac{1}{5}$. Find R_1 and R_2.

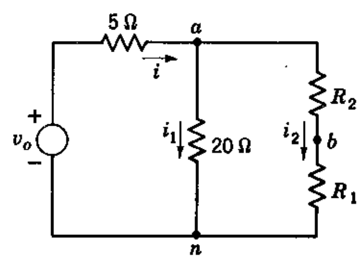

Fig. P3-30

3-31 In the circuit of Fig. P3-30, $i_1/i = \frac{3}{4}$ and $v_{bn}/v_o = \frac{1}{2}$. Find R_1 and R_2.

3-32 If $f(t) = (2p^3 + 4p^2 + 3p + 4)g(t)$, find $f(t)$ for (a) $g(t) = e^{-t}$; (b) $g(t) = te^{-t}$; (c) $e^{-t} + te^{-t}$; (d) $t^2 + 4t$.

3-33 Show that, if $f(t)$ is a forcing function for a response, then $Kf(t)$ is also a forcing function if K = const.

3-34 In the circuit of Fig. P3-34, show that (a) $i_1 = (2p + 2)v_2$; (b) $v_{an} = (4p^2 + 6p + 3)v_2$; (c) $i_0 = (4p^3 + 6p^2 + 5p + 2)v_2$; (d) $v_o = (8p^3 + 16p^2 + 16p + 7)v_2$; (e) the driving-point impedance $Z_{a'n}(p)$ is given by $Z_{a'n}(p) = (8p^3 + 16p^2 + 16p + 7)/(4p^3 + 6p^2 + 5p + 2)$.

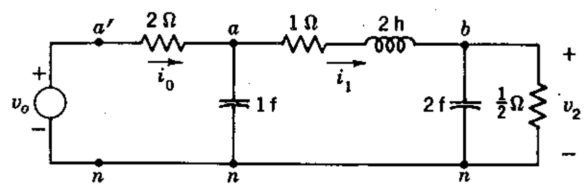

Fig. P3-34

3-35 In the circuit of Fig. P3-35: (a) Show that the differential equation that relates the response v_2 to the source function v_o is $(p^3R^3C^3 + 5p^2R^2C^2 + 6pRC + 1)v_2 = v_o$. (b) Use the result of part a to deduce the differential equation relating v_2 to v_o if each resistance R is replaced by inductance L.

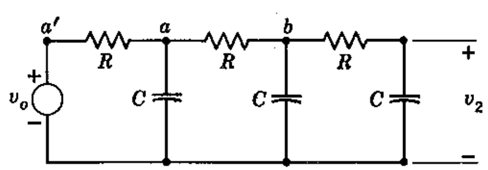

Fig. P3-35

3-37 In the circuit of Fig. P3-37, obtain the equilibrium equations relating i_2 and v_{ab} to the source function i_s.

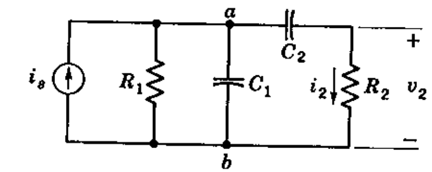

Fig. P3-37

3-38 In the circuit of Fig. 3-25, let C = 1 farad, L = 2 henrys, R_L = 1 ohm, and $R_C = \frac{1}{2}$ ohm. Obtain the equilibrium equation that relates v_{ab} to v.

3-39 In the circuit of Fig. P3-39, obtain the equilibrium equation that relates v_{ab} to $v_o (0 < a < 1)$.

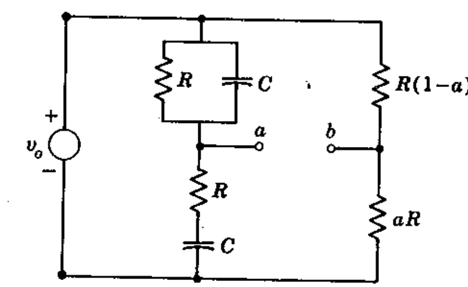

Fig. P3-39

3-40 In the circuit of Fig. P3-34 (whose driving-point impedance is as given in Prob. 3-34e), find the forcing functions for the response $i_0(t)$ if (a) $v_o(t) = 1$; (b) $v_o(t) = t$; (c) $v_o(t) = 1 + t + t^2$; (d) $v_o(t) = e^t$; (e) $v_o(t) = e^{-2t}$.

3-41 For the circuit of Fig. P3-37, let $R_2 = \frac{1}{9}R_1 = 1$ and $C_1 = \frac{1}{9}C_2 = 1$ and use the result to find the forcing function for the response v_2 if the source function is given by (a) $i_s = 12$; (b) $i_s = 12t$; (c) $i_s = 5e^{-2t}$.

3-42 In the circuit of Prob. 3-38, find the forcing function for the response $v_1(t)$ for the cases (a) $v(t) = 5 + 5t$; (b) $v(t) = e^{-t}$; (c) $v(t) = e^{-t/2}$.

3-43 Find the component of the response v_2 due to a constant source of value $v_o = V_o = 10$ in the circuit of (a) Fig. P3-34; (b) Fig. P3-35; (c) Fig. P3-39.

3-44 Find the component of the response due to the source $v = v_o(t) = 10e^{-2t}$ in the circuit of (a) Fig. P3-34 with v_2 as the response; (b) Fig. P3-35 with v_2 as the response and $RC = 2$; (c) Prob. 3-38 with v_1 as the response.

3-45 In a certain circuit the response v_2 is related to the source function v_o by the $v_2 = [(\frac{1}{2}p + 2)/(p + 5)]v_o$. Find the component of the response due to the source (a) $v_o = 10$; (b) $v_o = 5e^{-t}$; (c) $v_o = 10e^{-4t}$; (d) $v_o = 10e^{-t/4}$.

Analysis of electric circuits

SECOND EDITION

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