PROBLEM 1: Laplace & Transfer Function

30 points: Obtain the transfer function of a cascade filter designed by putting in cascade Circuit #2 and Circuit #3.

\[ \text{Circuit #1} \]

\[ R \]

\[ v_s(t) \]

\[ - \]

\[ C \]

\[ v_o(t) \]

\[ + \]

\[ v_s(t) \]

\[ + \]

\[ - \]

\[ \text{Circuit #2} \]

\[ C \]

\[ v_s(t) \]

\[ + \]

\[ R \]

\[ v_o(t) \]

\[ - \]

\[ \text{Circuit #3} \]

\[ L \]

\[ v_s(t) \]

\[ + \]

\[ R \]

\[ v_o(t) \]

\[ - \]

\[ \text{Circuit #4} \]

\[ R \]

\[ v_s(t) \]

\[ + \]

\[ L \]

\[ v_o(t) \]

\[ - \]
PROBLEM 1 – 30 points: We start with the $LR$-Filter, obtaining the differential equation which describes the dynamics of the filter and then proceed to take the Laplace transform on both sides of this equation.

K.V.L.:
\[ v_s(t) - v_l(t) - v_o(t) = 0; \quad \text{then,} \]
\[ v_l(t) = v_s(t) - v_o(t) = L \frac{d}{dt} i_l(t) \]

K.C.L.:
\[ i_l(t) - i_r(t) - i_o(t) = 0; \quad \text{then,} \quad i_l(t) = i_r(t) = \frac{v_o(t)}{R} \]
\[ v_s(t) - v_o(t) = \frac{L}{R} \frac{d}{dt} v_o(t); \quad v_o(t) + \frac{L}{R} \frac{d}{dt} v_o(t) = v_s(t) \]
\[ V_o(s) + \frac{L}{R} sV_o(s) = V_s(s); \quad \left(1 + \frac{L}{R} s\right)V_o(s) = V_s(s) \]
\[ H(s) = \frac{V_o(s)}{V_s(s)} = \frac{1}{1 + \frac{L}{R} s}; \quad \text{Thus,} \quad H_3(s) = \frac{\frac{R}{L}}{\frac{R}{L} + s} \]

We proceed with the \( CR \)-Filter, obtaining the differential equation which describes the dynamics of the filter and then proceed to take the Laplace transform on both sides.

\[ \textbf{K.V.L.:} \]
\[ v_s(t) - v_c(t) - v_o(t) = 0; \quad \text{then,} \quad v_c(t) = v_s(t) - v_o(t) \]

\[ \textbf{K.C.L.:} \]
\[ i_c(t) - i_r(t) - i_o(t) = 0; \quad \text{then,} \]
\[ i_c(t) = i_r(t) = \frac{v_o(t)}{R} = C \frac{d}{dt} v_c(t) \]
\[ \frac{v_o(t)}{R} = C \frac{d}{dt} \{ v_s(t) - v_o(t) \}; \quad v_o(t) + RC \frac{d}{dt} v_o(t) = RC \frac{d}{dt} v_s(t) \]
\[ V_o(s) + RC s V_o(s) = RC s V_s(s); \quad (1 + RC s) V_o(s) = V_s(s) \]

\[ H(s) = \frac{V_o(s)}{V_s(s)} = \frac{RC s}{1 + RC s}; \quad \text{Thus,} \quad H_2(s) = \frac{s}{\frac{1}{RC} + s} \]

We proceed to use the Laplace transform’s convolution theorem to obtain the final result:
\[ H(s) = \frac{V_o(s)}{V_s(s)} = H_3(s) \bullet H_2(s) = \left( \frac{\frac{R}{L}}{\frac{R}{L} + s} \right) \bullet \left( \frac{s}{\frac{1}{RC} + s} \right) = \frac{\frac{R}{L} s}{\frac{R}{L} + s} \left( \frac{1}{\frac{1}{RC} + s} \right) \]

Please, provide clear and concrete answers.
No credits will be given for partial solutions.
DO NOT USE PROGRAMMABLE CALCULATORS OR SMART PHONES.
Problem Two: Ideal Low-Pass linear Phase Filters

The spectrum of the impulse response function \( T\{\delta(t)\} = h_L(t) \) is:

\[
H_L(f) = \begin{cases} 
  e^{-j\frac{\pi}{2}f}, & |f| \leq 120 \\
  0, & |f| > 120 
\end{cases}
\]

a) (30 points) Get \( T\{\cos(2\pi 50t) + \cos(\pi 100t) + \cos(2\pi 180t)\} \)

b) (40 points) Get the output signal of the filter if the input signal is a periodic symmetric square wave which has a duty cycle of one half and a fundamental frequency of 50 Hz.

**SOLUTION:**

We use linearity and the Fourier transform theorem on

\[
T\{\cos(2\pi 50t) + \cos(\pi 100t) + \cos(2\pi 180t)\}
\]

as follows:

\[
T\{\cos 2\pi f dt\} = T\left\{\frac{1}{2} e^{+ j 2\pi f dt} + \frac{1}{2} e^{- j 2\pi f dt} \right\} = \frac{1}{2} T\{e^{+ j 2\pi f dt}\} + \frac{1}{2} T\{e^{- j 2\pi f dt}\}
\]

Remember that:

\[
T\{e^{+ j 2\pi f dt}\} = \int_{-\infty}^{+\infty} h(\tau) e^{+ j 2\pi f (t-\tau)} d\tau = e^{+ j 2\pi f dt} \int_{-\infty}^{+\infty} h(\tau) e^{- j 2\pi f \tau} d\tau
\]

Thus,

\[
T\{e^{+ j 2\pi f dt}\} = e^{+ j 2\pi f dt} \int_{-\infty}^{+\infty} h(\tau) e^{- j 2\pi f \tau} d\tau = H(f_d) e^{+ j 2\pi f dt}
\]

and, we get the output of the system with a cosine function as input:

\[
T\{\cos 2\pi f dt\} = \frac{1}{2} H(f_d) e^{+ j 2\pi f dt} + \frac{1}{2} H(-f_d) e^{- j 2\pi f dt}
\]
We proceed to evaluate the frequency response function:

For \( f_d = 50 \), \( H_L(f_d) = e^{-j \frac{2\pi}{30}(50)} \); \( H_L(-f_d) = e^{-j \frac{2\pi}{30}(-50)} \)

For \( f_d = 50 \), \( H_L(f_d) = e^{-j \frac{2\pi}{30}(50)} \); \( H_L(-f_d) = e^{-j \frac{2\pi}{30}(-50)} \)

For \( f_d = 180 \), \( H_L(f_d) = 0 \) since \( f_d > f_m \), and \( f_m = 120 \).

Thus, we obtain the following answer:

\[
T \{ \cos 2\pi f_d t \} = e^{-j \frac{2\pi}{30}(50)} e^{j 2\pi f_d t} + e^{j \frac{2\pi}{30}(50)} e^{-j 2\pi f_d t}
\]

Thus, the system was able to filter to remove the signal with the highest frequency, at \( f_d = 180 \text{ Hz} \), and allowed to pass the lower frequencies, \( f_d = 50 \text{ Hz} \), twice.

To get the output signal of the filter if the input signal is a periodic symmetric square wave which has a duty cycle of one half and a fundamental frequency of 50 Hz, we proceed as follows:

A periodic symmetric square wave with duty cycle \( C_D = \frac{1}{2} \) and fundamental frequency \( F_p = 50 \text{ Hz} \) may be represented in terms of complex exponential Fourier series as follows:

\[
s(t) = \sum_{n=-\infty}^{n=+\infty} C_n e^{j 2\pi n F_p t}, \text{ where}
\]

\[
C_n = \frac{1}{T_p} \int_{-\infty}^{+\infty} s(t) e^{-j 2\pi n F_p t} dt
\]
The output of the filter is given in the frequency domain by

\[ Y(f) = H_L(f) \cdot S(f), \quad \text{where} \]

\[ S(f) = \mathcal{F}\{s(t)\} = \sum_{n=-\infty}^{n=+\infty} C_n e^{+j2\pi nF_p t} = \sum_{n=-\infty}^{n=+\infty} C_n F_p e^{+j2\pi nF_p t} \]

Thus,

\[ S(f) = \sum_{n=-\infty}^{n=+\infty} C_n \delta(f - nF_p) \]

Since the cutoff frequency of the filter is \( f_m = 120 \), and the fundamental frequency is \( F_p = 50 \), only two harmonics are allowed to pass through the filter, the fundamental or first harmonic at \( F_p = 50 \), and the second harmonic at \( 2F_p = 100 \).

Thus, we get the following result

\[ Y(f) = H_L(f) \cdot \left( \sum_{n=-2}^{n=+2} C_n \delta(f - nF_p) \right) \]

In the time domain we get

\[ y(t) = \left( \sum_{n=-2}^{n=+2} H_L(nF_p)C_n e^{+j2\pi nF_p t} \right) \]