

# 24

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## Butterworth Filters

To illustrate some of the ideas developed in Lecture 23, we introduce in this lecture a simple and particularly useful class of filters referred to as *Butterworth filters*. Filters in this class are specified by two parameters, the cutoff frequency and the filter order. The frequency response of these filters is monotonic, and the sharpness of the transition from the passband to the stopband is dictated by the filter order. For continuous-time Butterworth filters, the poles associated with the square of the magnitude of the frequency response are equally distributed in angle on a circle in the  $s$ -plane, concentric with the origin and having a radius equal to the cutoff frequency. When the cutoff frequency and the filter order have been specified, the poles characterizing the system function are readily obtained. Once the poles are specified, it is straightforward to obtain the differential equation characterizing the filter.

In this lecture, we illustrate the design of a discrete-time filter through the use of the impulse-invariant design procedure applied to a Butterworth filter. The filter specifications are given in terms of the discrete-time frequency variable and then mapped to a corresponding set of specifications for the continuous-time filter. A Butterworth filter meeting these specifications is determined. The resulting continuous-time system function is then mapped to the desired discrete-time system function.

A limitation on the use of impulse invariance as a design procedure for discrete-time systems is the requirement that the continuous-time filter be bandlimited to avoid aliasing. An alternative procedure, called the *bilinear transformation*, corresponds to a mapping of the entire imaginary axis in the  $s$ -plane to once around the unit circle. Consequently, there is no aliasing introduced by this procedure. However, since the imaginary axis has infinite length and the unit circle has a finite circumference, by necessity there must be a nonlinear distortion in mapping between the two frequency axes. The use of the bilinear transformation is therefore limited to mapping filters that are approximately piecewise constant. For filters of this type, the inherent nonlinear distortion in the frequency axis is easily accommodated by prewarping the critical frequencies (e.g., passband edge and stopband edge) prior to carrying out the design of the associated continuous-time system. This procedure is illustrated with the design of a Butterworth filter.

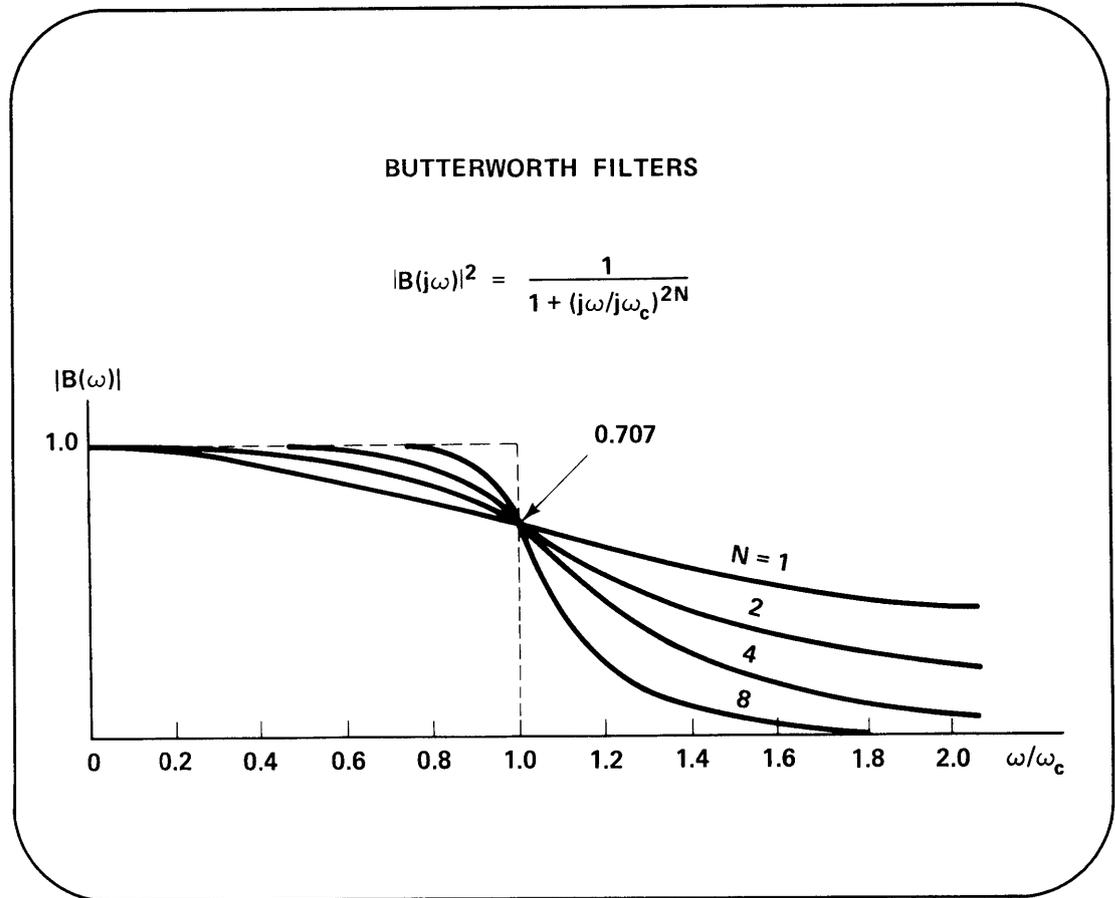
### Suggested Reading

Section 6.5, The Class of Butterworth Frequency-Selective Filters, pages 422–428

Section 9.7.3, Butterworth Filters, pages 611–614

Section 10.8.3, The Bilinear Transformation, pages 665–667

**TRANSPARENCY**  
**24.1**  
 Frequency response  
 for the class of  
 Butterworth filters.



**TRANSPARENCY**  
**24.2**  
 The square of the  
 magnitude of the  
 frequency response  
 and the system  
 function for  
 Butterworth filters.

$$|B(j\omega)|^2 = \frac{1}{1 + (j\omega/j\omega_c)^{2N}}$$

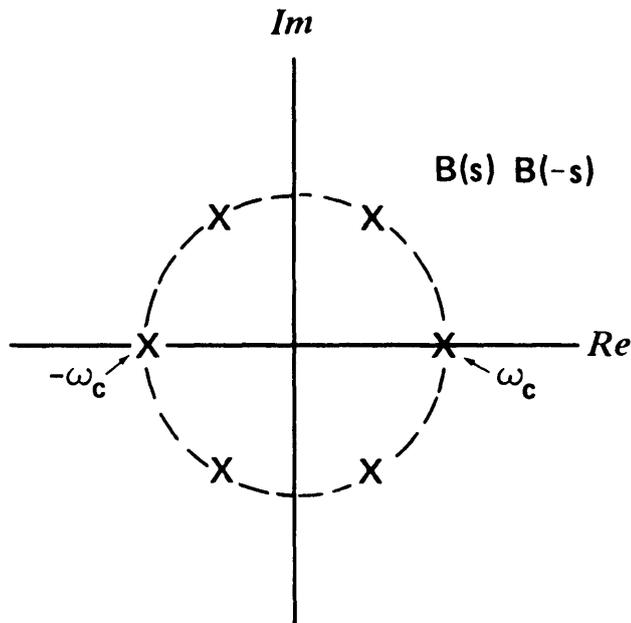
$$B(j\omega) B(-j\omega) \rightarrow B(s) B(-s)$$

$$B(s) B(-s) = \frac{1}{1 + (s/j\omega_c)^{2N}}$$

poles of  $B(s) B(-s)$  at

$$s = (-1)^{1/2N} j\omega_c$$

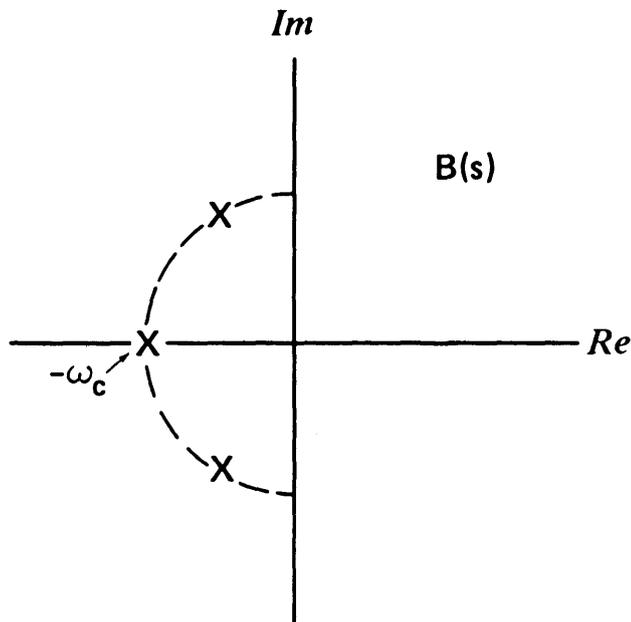
$$B(s) B(-s) = \frac{1}{1 + (s/j\omega_c)^{2N}}$$



**TRANSPARENCY 24.3**

Transparencies 24.3 and 24.4 show a pole-zero plot associated with the Butterworth filter. Shown here is the pole-zero pattern for the square of the magnitude of the frequency response for Butterworth filters.

$$B(s) B(-s) = \frac{1}{1 + (s/j\omega_c)^{2N}}$$

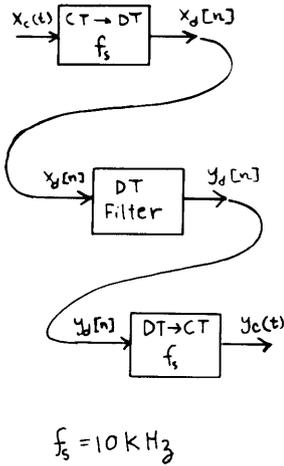


**TRANSPARENCY 24.4**

The pole-zero plot for the system function for a Butterworth filter. Since we restrict  $B(s)$  to correspond to a stable, causal filter, its poles must all be in the left half of the  $s$ -plane.

MARKERBOARD

24.1 (a)

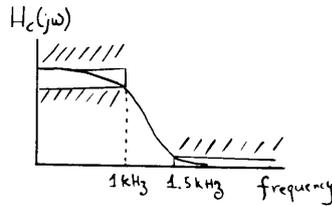


$f_s = 10 \text{ kHz}$

Sampling Rate = 10 kHz

Desired Specifications on  $H_c(j\omega)$ :

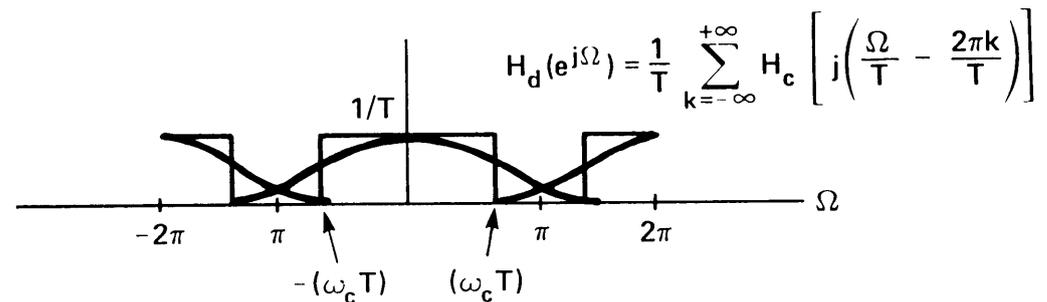
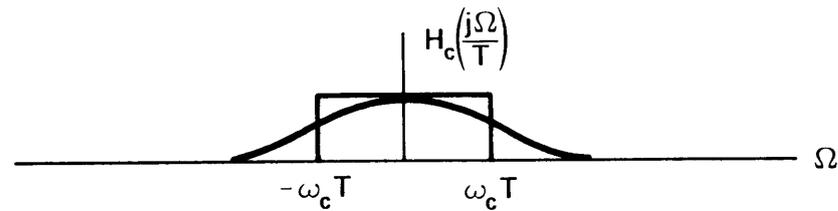
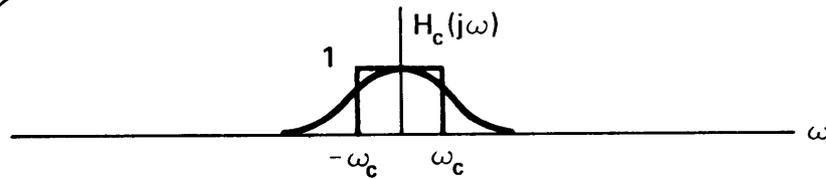
- $20 \log_{10} |H_c(j\omega)| \geq -1 \text{ (db)}$  at  $\omega = 2\pi \cdot 1 \text{ kHz}$
- $20 \log_{10} |H_c(j\omega)| \leq -15 \text{ (db)}$  at  $\omega = 2\pi \cdot 1.5 \text{ kHz}$



Corresponding Specifications on  $H_d(e^{j\Omega})$

- $10 \text{ kHz} \rightarrow 2\pi$
- $1 \text{ kHz} \rightarrow 0.2\pi$
- $1.5 \text{ kHz} \rightarrow 0.3\pi$
- $20 \log_{10} |H_d(e^{j\Omega})| \geq -1$  at  $\Omega = 0.2\pi$
- $20 \log_{10} |H_d(e^{j\Omega})| \leq -15$  at  $\Omega = 0.3\pi$
- $h_d[n] = h_c(nT)$
- $T = ?$

TRANSPARENCY  
24.5  
Spectra illustrating impulse invariance.



$$h_d[n] = h_b(nT) \quad T=1$$

$$20 \log_{10} |B(j0.2\pi)| \geq -1$$

$$20 \log_{10} |B(j0.3\pi)| \leq -15$$

$$|B(j\omega)|^2 = \frac{1}{1 + (j\omega/j\omega_c)^{2N}}$$

equality for  $N = 5.88$   
 $\omega_c = 0.7047$

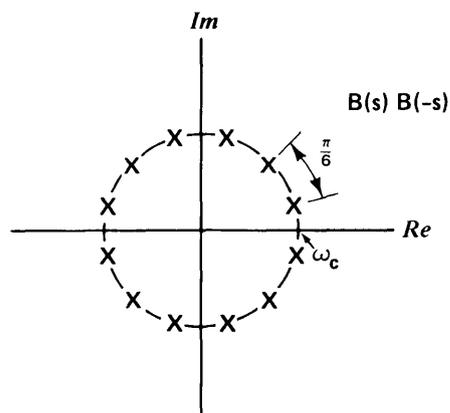
choose  $N = 6$   
 $\omega_c = 0.703$

**TRANSPARENCY**  
**24.6**

Specifications on a discrete-time filter to be obtained from a continuous-time Butterworth filter using impulse invariance.

$$N=6$$

$$\omega_c = 0.703$$



1. Determine  $B(s)$

2. Partial fraction expansion

$$B(s) = \sum_k \frac{A_k}{s - s_k}$$

3.  $H_d(z) = \sum_k \frac{A_k}{1 - e^{s_k} z^{-1}}$

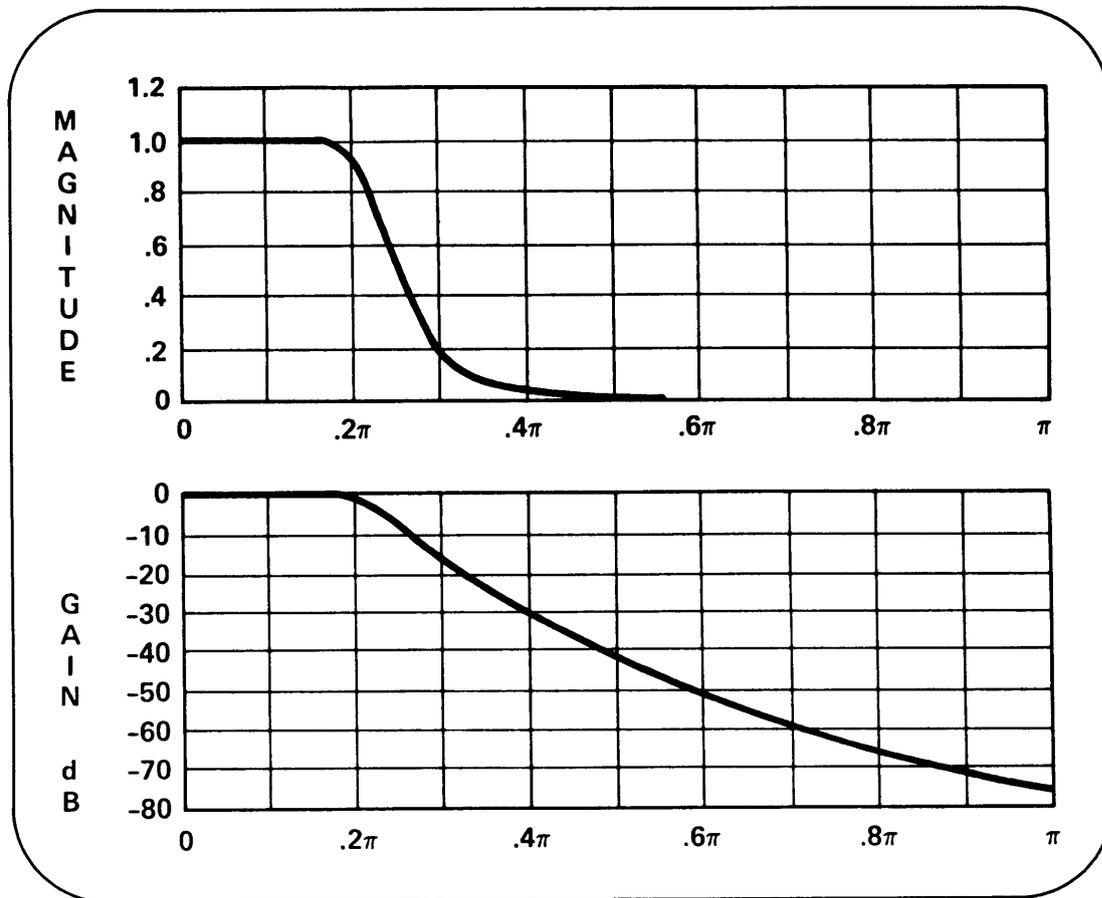
**TRANSPARENCY**  
**24.7**

Pole-zero plot associated with the Butterworth filter to be mapped to the desired discrete-time filter and a summary of the steps in the procedure.

**TRANSPARENCY**

24.8

Frequency response of the discrete-time Butterworth filter designed using impulse invariance.



**BILINEAR TRANSFORMATION**

**TRANSPARENCY**

24.9

The bilinear transformation for mapping from continuous-time to discrete-time filters.

$$H_c(s) \longrightarrow H_d(z)$$

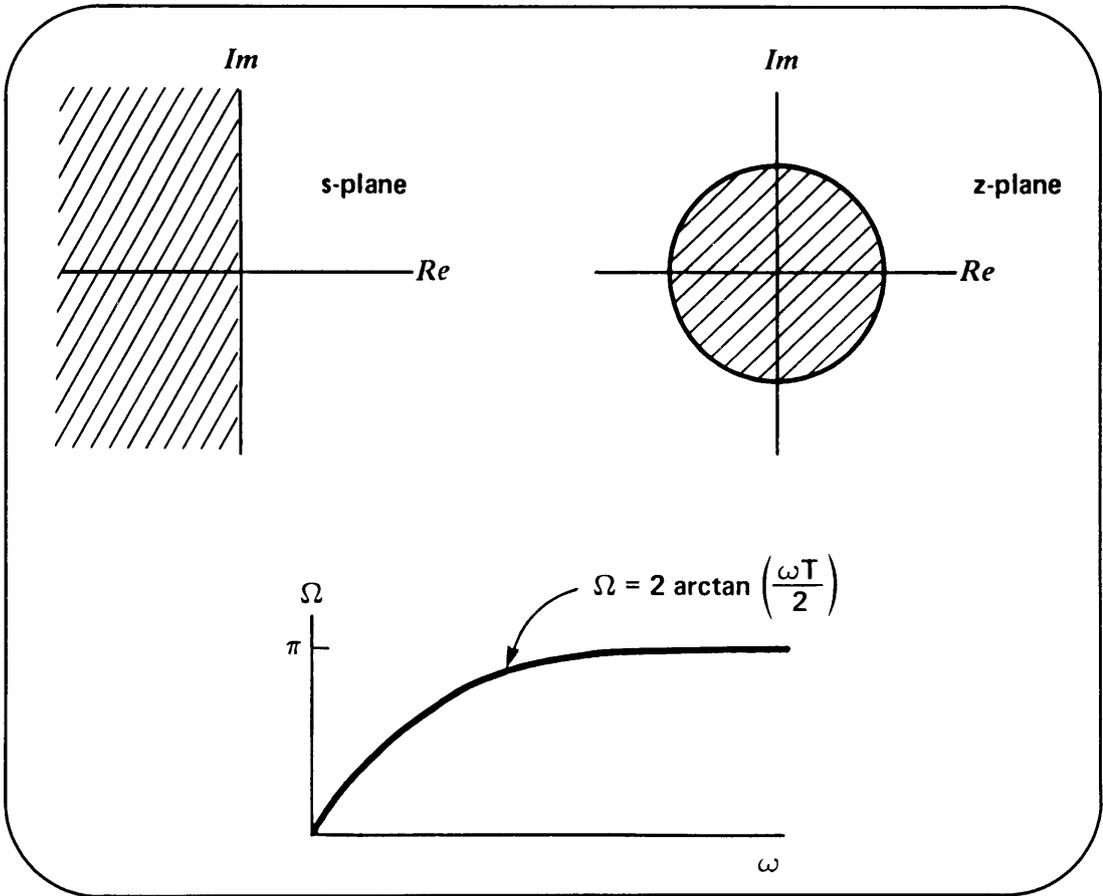
$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$z = \frac{1 + (T/2)s}{1 - (T/2)s}$$

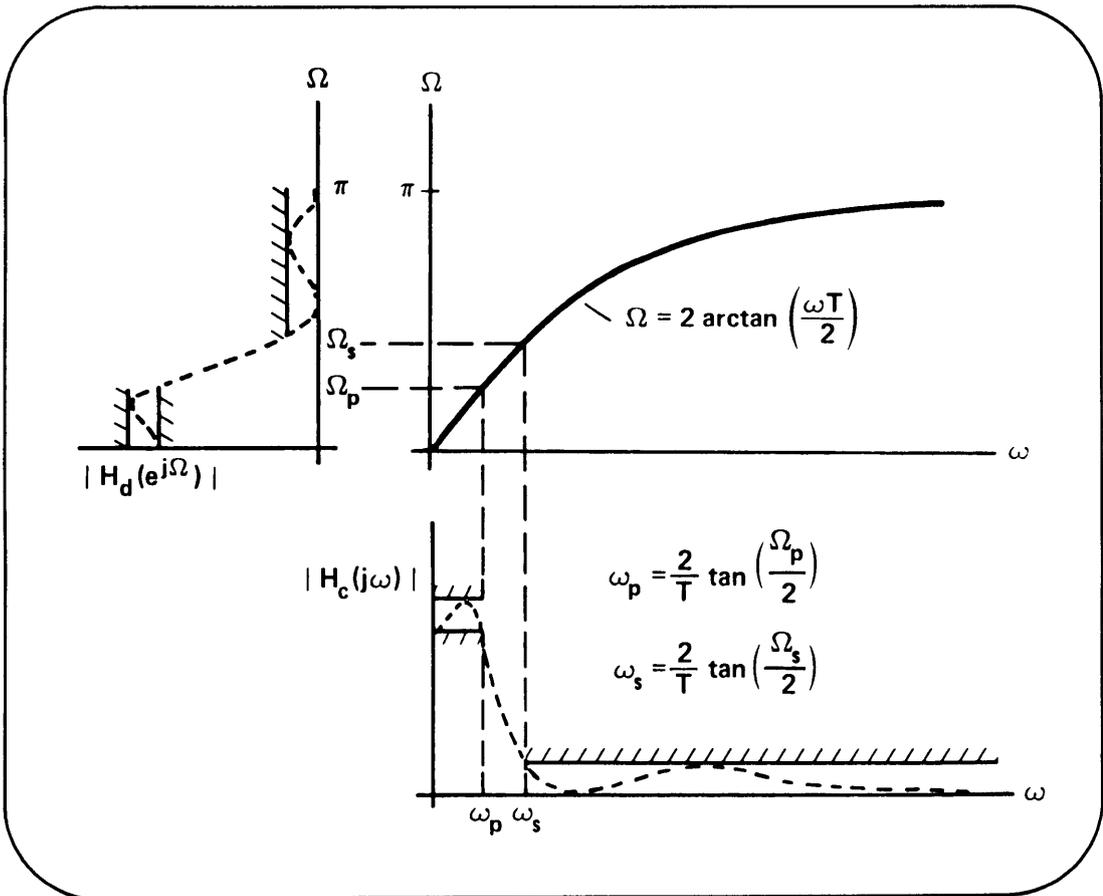
For  $z = e^{j\Omega}$

$$s = \frac{2}{T} \left[ \frac{1 - e^{-j\Omega}}{1 + e^{-j\Omega}} \right] = \frac{2}{T} j \tan \frac{\Omega}{2} = j\omega$$

$$\omega = \frac{2}{T} \tan \frac{\Omega}{2}$$

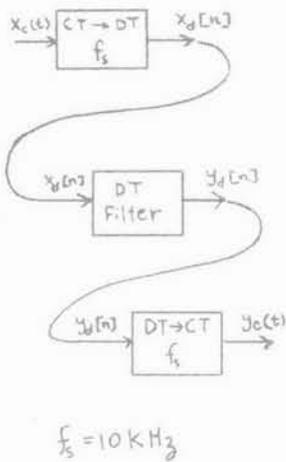


**TRANSPARENCY 24.10**  
Mapping from the continuous-time frequency axis to the discrete-time frequency axis resulting from the bilinear transformation.



**TRANSPARENCY 24.11**  
Illustration of the effect of the bilinear transformation on the approximation to a lowpass filter.

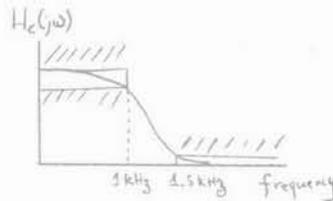
**MARKERBOARD**  
24.1 (b)



Sampling Rate = 10 kHz

Desired Specifications on  $H_c(j\omega)$ :

- $20 \log_{10} |H_c(j\omega)| \geq -1$  (db) at  $\omega = 2\pi \cdot 1 \text{ kHz}$
- $20 \log_{10} |H_c(j\omega)| \leq -15$  (db) at  $\omega = 2\pi \cdot 1.5 \text{ kHz}$



Corresponding Specifications on  $H_d(e^{j\Omega})$

- $10 \text{ kHz} \rightarrow 2\pi$
- $1 \text{ kHz} \rightarrow 0.2\pi$
- $1.5 \text{ kHz} \rightarrow 0.3\pi$
- $20 \log_{10} |H_d(e^{j\Omega})| \geq -1$  at  $\Omega = 0.2\pi$
- $20 \log_{10} |H_d(e^{j\Omega})| \leq -15$  at  $\Omega = 0.3\pi$
- $h_d[n] = h_c(nT)$
- $T = ?$

**TRANSPARENCY**

24.12

Determination of the parameters for a Butterworth filter to be mapped to a discrete-time filter with the specifications in Transparency 24.6.

$$\Omega = 2 \arctan \left( \frac{\omega T}{2} \right) \quad T=1$$

$$\omega = 2 \tan \left( \frac{\Omega}{2} \right)$$

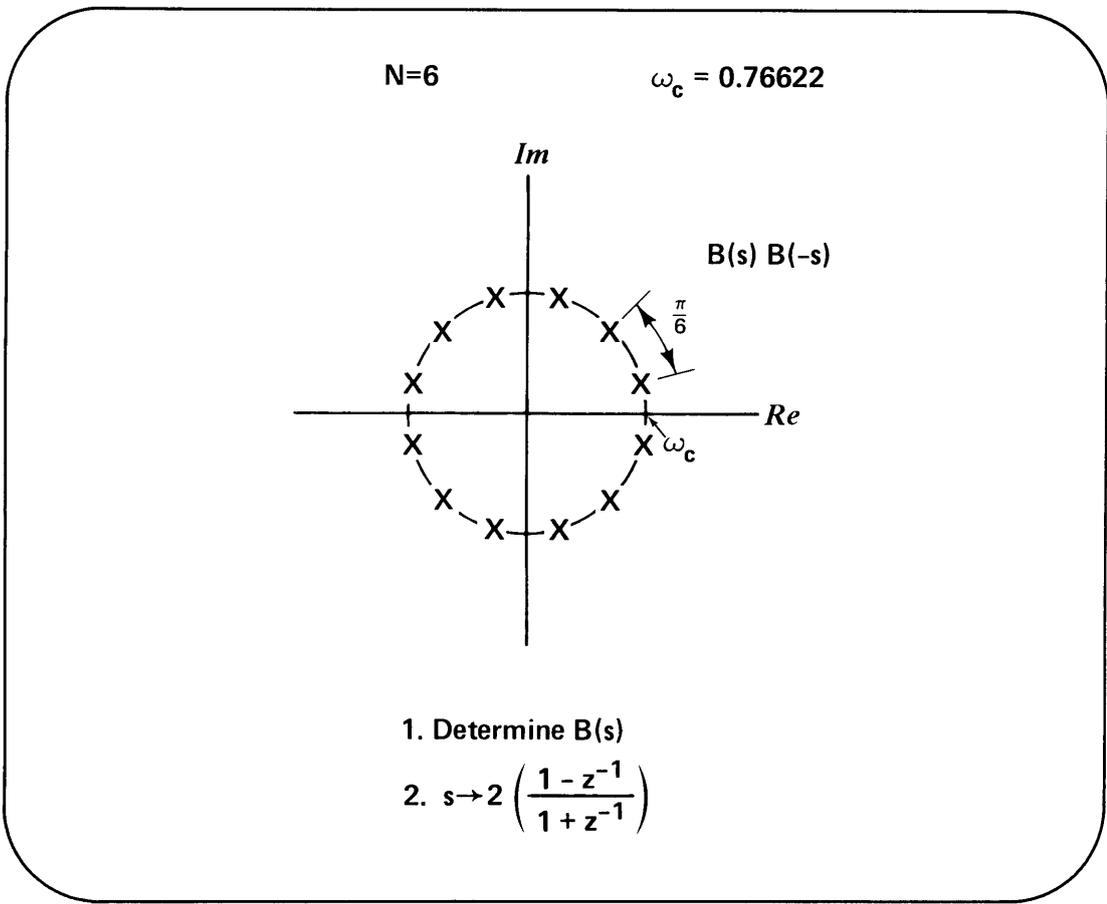
$$20 \log_{10} \left| G \left( j 2 \tan \left( \frac{0.2\pi}{2} \right) \right) \right| \geq -1$$

$$20 \log_{10} \left| G \left( j 2 \tan \left( \frac{0.3\pi}{2} \right) \right) \right| \leq -15$$

equality for  $N = 5.3$

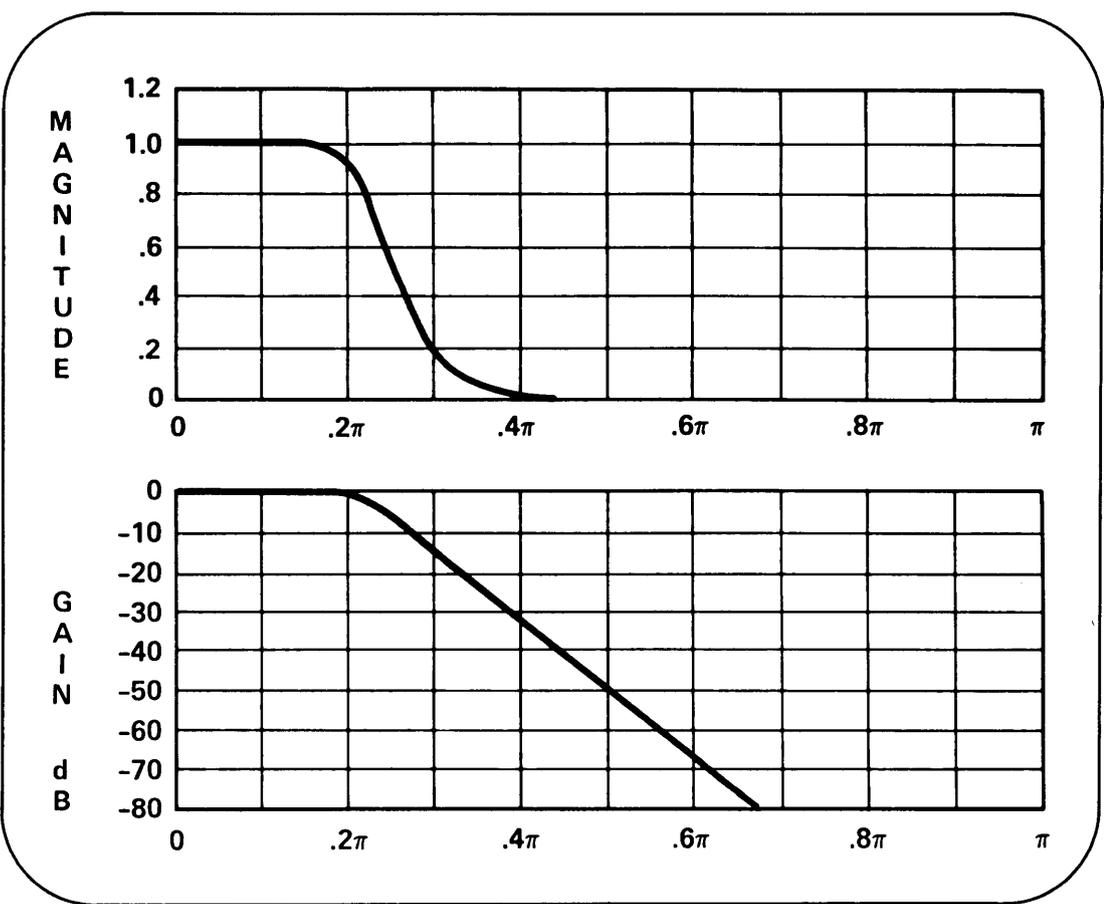
choose  $N = 6$

$$\omega_c = 0.76622$$



**TRANSPARENCY 24.13**

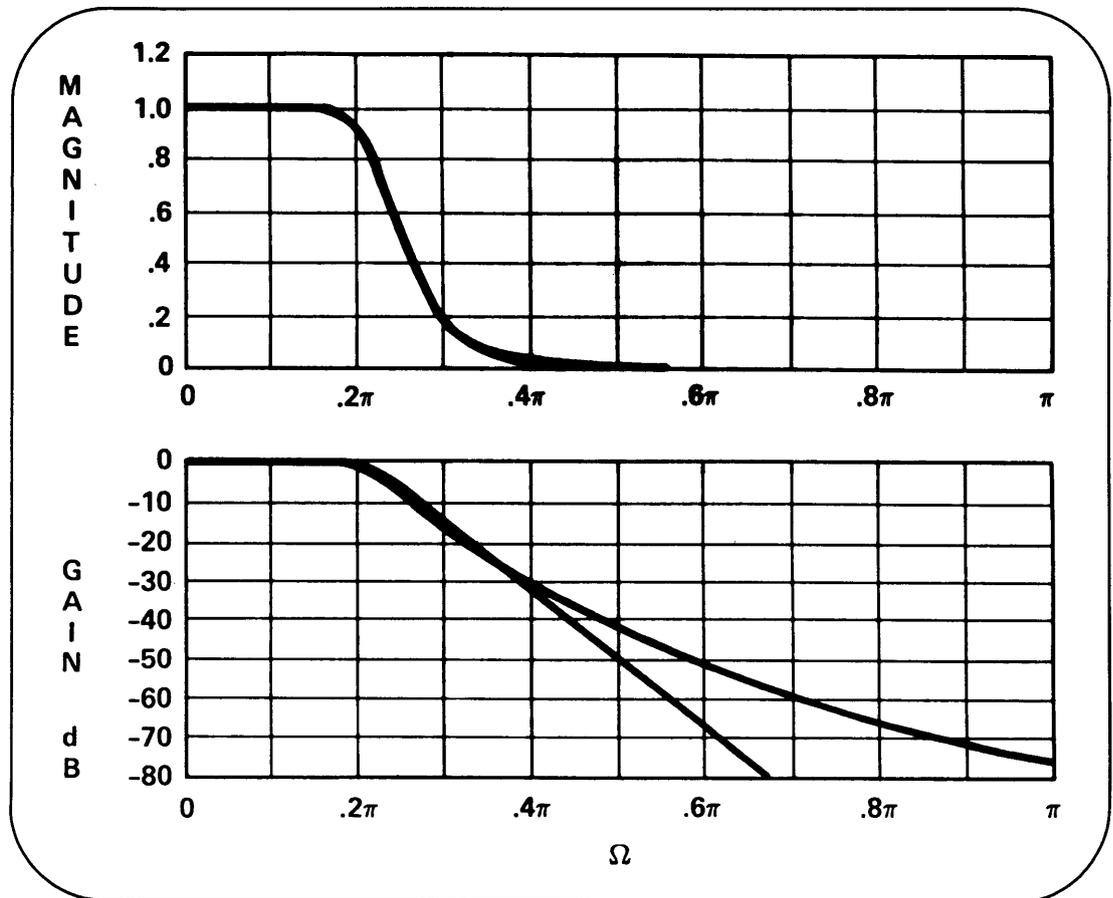
The pole-zero plot associated with the squared magnitude function for the desired Butterworth filter and the steps involved in the determination of the discrete-time filter.



**TRANSPARENCY 24.14**

Frequency response for the discrete-time filter obtained by mapping a Butterworth filter to a digital filter through the bilinear transformation.

**TRANSPARENCY**  
**24.15**  
 Comparison of the frequency responses of digital filters obtained through impulse invariance and through the bilinear transformation.



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