

23 Mapping Continuous-Time Filters to Discrete-Time Filters

Recommended Problems

P23.1

For each of the following sequences, determine the associated z -transform, including the ROC. Make use of properties of the z -transform wherever possible.

- (a) $x_1[n] = (\frac{1}{2})^n u[n]$
- (b) $x_2[n] = (-3)^n u[-n]$
- (c) $x_3[n] = x_1[n] + x_2[n]$
- (d) $x_4[n] = x_1[n - 5]$
- (e) $x_5[n] = x_1[n + 5]$
- (f) $x_6[n] = (\frac{1}{3})^n u[n]$
- (g) $x_7[n] = x_1[n] * x_6[n]$

P23.2

A causal LTI system is described by the difference equation

$$y[n] - 3y[n - 1] + 2y[n - 2] = x[n]$$

- (a) Find $H(z) = Y(z)/X(z)$. Plot the poles and zeros and indicate the ROC.
- (b) Find the unit sample response. Is the system stable? Justify your answer.
- (c) Find $y[n]$ if $x[n] = 3^n u[n]$.
- (d) Determine the system function, associated ROC, and impulse response for all LTI systems that satisfy the preceding difference equation but are not causal. In each case, specify whether the corresponding system is stable.

P23.3

Carry out the proof for the following properties from Table 10.1 of the text (page 654).

- (a) 10.5.2
- (b) 10.5.3
- (c) 10.5.6 [*Hint*: Consider

$$\left. \frac{dX(z)}{dz} = \frac{d}{dz} \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right]$$

P23.4

Consider the second-order system with the pole-zero plot given in Figure P23.4. The poles are located at $z = re^{j\theta}$, $z = re^{-j\theta}$, and $H(1) = 1$.

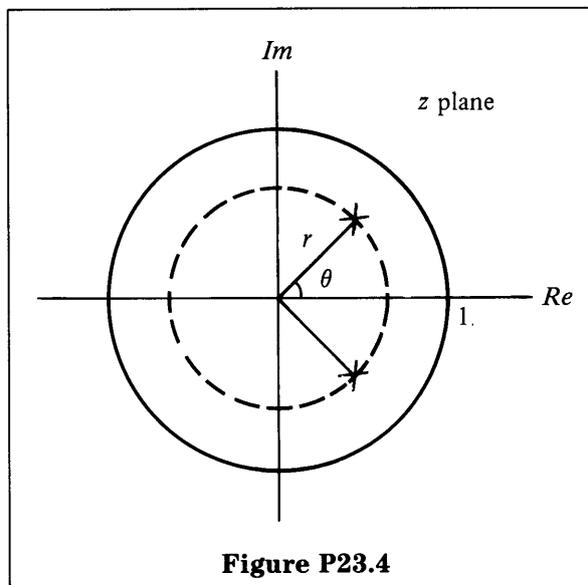


Figure P23.4

- (a) Sketch $|H(e^{j\theta})|$ as θ is kept constant at $\pi/4$ and with $r = 0.5, 0.75,$ and 0.9 .
- (b) Sketch $H(e^{j\theta})$ as r is kept constant at $r = 0.75$ and with $\theta = \pi/4, 2\pi/4,$ and $3\pi/4$.

P23.5

Consider the system function

$$H(z) = \frac{z}{(z - \frac{1}{3})(z - 2)}$$

- (a) Sketch the pole-zero locations.
- (b) Sketch the ROC assuming the system is causal. Is the system stable?
- (c) Sketch the ROC assuming the system is stable. Is the system causal?
- (d) Sketch the remaining possible ROC. Is the corresponding system either stable or causal?

P23.6

Consider the continuous-time LTI system described by the following equation:

$$\frac{d^2y(t)}{dt^2} + 5 \frac{dy(t)}{dt} + 6y(t) = x(t) + 2 \frac{dx(t)}{dt}$$

- (a) Determine the system function $H_c(s)$ and the impulse response $h_c(t)$.
- (b) Determine the system function $H_d(z)$ of a discrete-time LTI system obtained from $H_c(s)$ through impulse invariance.
- (c) For $T = 0.01$ determine the impulse response associated with $H_d(z), h_d[n]$.
- (d) Verify that $h_d[n] = h_c(nT)$ for all n .

Optional Problems

P23.7

Consider a continuous-time LTI filter described by the following differential equation:

$$\frac{dy(t)}{dt} + 0.5y(t) = x(t)$$

A discrete-time filter is obtained by replacing the derivative by a first forward difference to obtain the difference equation

$$\frac{y[n+1] - y[n]}{T} + 0.5y[n] = x[n]$$

Assume that the resulting system is causal.

- Determine and sketch the magnitude of the frequency response of the continuous-time filter.
- Determine and sketch the magnitude of the frequency response of the discrete-time filter for $T = 2$.
- Determine the range of values of T (if any) for which the discrete-time filter is unstable.

P23.8

Consider an even sequence $x[n]$ (i.e., $x[n] = x[-n]$) with rational z -transform $X(z)$.

- From the definition of the z -transform show that

$$X(z) = X\left(\frac{1}{z}\right)$$

- From your result in part (a), show that if a pole (or a zero) of $X(z)$ occurs at $z = z_0$, then a pole (or a zero) must also occur at $z = 1/z_0$.
- Verify the result in part (b) for each of the following sequences:
 - $\delta[n+1] + \delta[n-1]$
 - $\delta[n+1] - \frac{5}{2}\delta[n] + \delta[n-1]$
- Consider a real-valued sequence $y[n]$ with rational z -transform $Y(z)$.
 - Show that $Y(z) = Y^*(z^*)$.
 - From part (i) show that if a pole (or a zero) of $Y(z)$ occurs at $z = z_0$, then a pole (or a zero) must also occur at $z = z_0^*$.
- By combining your result in part (b) with that in part (d), show that for a real, even sequence, if there is a pole (or a zero) of $H(z)$ at $z = \rho e^{j\theta}$, then there is also a pole (or a zero) of $H(z)$ at $z = \rho e^{-j\theta}$, at $z = (1/\rho)e^{j\theta}$, and at $z = (1/\rho)e^{-j\theta}$.

P23.9

In Section 10.5.5 of the text we stated the convolution property for the z -transform. To prove this property, we begin with the convolution sum expressed as

$$x_3[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{+\infty} x_1[k]x_2[n - k] \quad (\text{P23.9-1})$$

(a) By taking the z -transform of eq. (P23.9-1) and using eq. (10.3) of the text (page 630), show that

$$X_3(z) = \sum_{k=-\infty}^{+\infty} x_1[k]\hat{X}_2(z),$$

where $\hat{X}_2(z) = Z\{x_2[n - k]\}$.

(b) Using the result in part (a) and the time-shifting property of z -transforms, show that

$$X_3(z) = X_2(z) \sum_{k=-\infty}^{+\infty} x_1[k]z^{-k}$$

(c) From part (b), show that

$$X_3(z) = X_1(z)X_2(z).$$

P23.10

Consider a signal $x[n]$ that is absolutely summable and its associated z -transform $X(z)$. Show that the z -transform of $y[n] = x[n]u[n]$ can have poles only at the poles of $X(z)$ that are inside the unit circle.

P23.11

Let $h_c(t)$, $s_c(t)$, and $H_c(s)$ denote the impulse response, step response, and system function, respectively, of a continuous-time, linear, time-invariant filter.

Let $h_d[n]$, $s_d[n]$, and $H_d(z)$ denote the unit sample response, step response, and system function, respectively, of a discrete-time, linear, time-invariant filter.

(a) If $h_d[n] = h_c(nT)$, does $s_d[n] = \sum_{k=-\infty}^n h_c(kT)$?

(b) If $s_d[n] = s_c(nT)$, does $h_d[n] = h_c(nT)$?

P23.12

Consider a continuous-time filter with input $x_c(t)$ and output $y_c(t)$ that is described by a linear constant-coefficient differential equation of the form

$$\sum_{k=0}^N a_k \frac{d^k y_c(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x_c(t)}{dt^k} \quad (\text{P23.12-1})$$

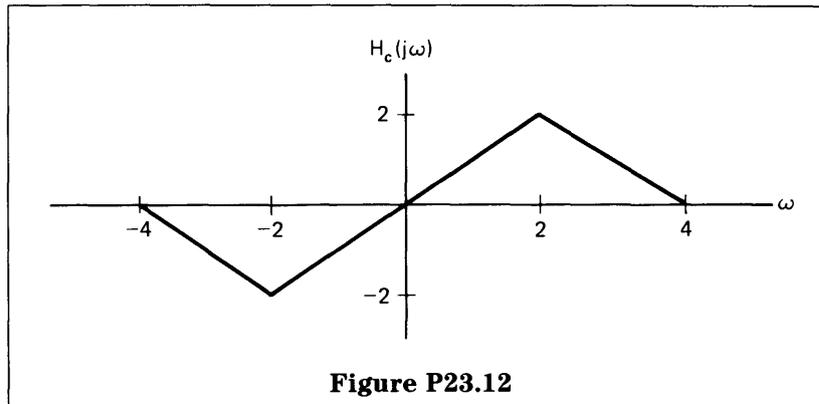
The filter is to be mapped to a discrete-time filter with input $x[n]$ and output $y[n]$ by replacing derivatives with central differences. Specifically, let $\nabla^{(k)}\{x[n]\}$ denote the k th central difference of $x[n]$, defined as follows:

$$\begin{aligned}\nabla^{(0)}\{x[n]\} &= x[n] \\ \nabla^{(1)}\{x[n]\} &= \left[\frac{x[n+1] - x[n-1]}{2} \right] \\ \nabla^{(k)}\{x[n]\} &= \nabla^{(1)}\{\nabla^{(k-1)}\{x[n]\}\}\end{aligned}$$

The difference equation for the digital filter obtained from the differential equation (P23.12-1) is then

$$\sum_{k=0}^N a_k \nabla^{(k)}\{y[n]\} = \sum_{k=0}^M b_k \nabla^{(k)}\{x[n]\}$$

- (a) If the transfer function of the continuous-time filter is $H_c(s)$ and if the transfer function of the corresponding discrete-time filter is $H_d(z)$, determine how $H_d(z)$ is related to $H_c(s)$.
- (b) For the continuous-time frequency response $H_c(j\omega)$, as indicated in Figure P23.12-1, sketch the discrete-time frequency response $H_d(e^{j\Omega})$ that would result from the mapping determined in part (a).



- (c) Assume that $H_c(s)$ corresponds to a causal stable filter. If the region of convergence of $H_d(z)$ is specified to include the unit circle, will $H_d(z)$ necessarily correspond to a *causal* filter?

P23.13

In discussing impulse invariance in Section 10.8.1 of the text, we considered $H_c(s)$ to be of the form of eq. (10.84) of the text with only first-order poles. In this problem we consider how the presence of a second-order pole in eq. (10.84) would be reflected in eq. (10.87) of the text. Toward this end, consider $H_c(s)$ to be

$$H_c(s) = \frac{A}{(s - s_0)^2}$$

- (a) By referring to Table 9.2 of the text (page 604), determine $h_c(t)$. (Assume causality.)

- (b) Determine $h_d[n]$ defined as $h_d[n] = h_c(nT)$.
- (c) By referring to Table 10.2 of the text (page 655), determine $H_d(z)$, the z -transform of $h_d[n]$.
- (d) Determine the system function and pole-zero pattern for the discrete-time system obtained by applying impulse invariance to the following continuous-time system:

$$H_c(s) = \frac{1}{(s + 1)(s + 2)^2}$$

MIT OpenCourseWare
<http://ocw.mit.edu>

Resource: Signals and Systems
Professor Alan V. Oppenheim

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.