22 The z-Transform

Recommended Problems

P22.1

An LTI system has an impulse response $h[n]$ for which the z-transform is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

(a) Plot the pole-zero pattern for $H(z)$.

(b) Using the fact that signals of the form z^n are eigenfunctions of LTI systems, determine the system output for all n if the input $x[n]$ is

$$x[n] = (\tfrac{3}{4})^n + 3(2)^n$$

P22.2

Consider the sequence $x[n] = 2^n u[n]$.

(a) Is $x[n]$ absolutely summable?

(b) Does the Fourier transform of $x[n]$ converge?

(c) For what range of values of r does the Fourier transform of the sequence $r^{-n}x[n]$ converge?

(d) Determine the z-transform $X(z)$ of $x[n]$, including a specification of the ROC.

(e) $X(z)$ for $z = 3e^{j\Omega}$ can be thought of as the Fourier transform of a sequence $x_1[n]$, i.e.,

$$2^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z),$$
$$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X(3e^{j\Omega}) = X_1(e^{j\Omega})$$

Determine $x_1[n]$.

P22.3

Shown in Figure P22.3 is the pole-zero plot for the z-transform $X(z)$ of a sequence $x[n]$.

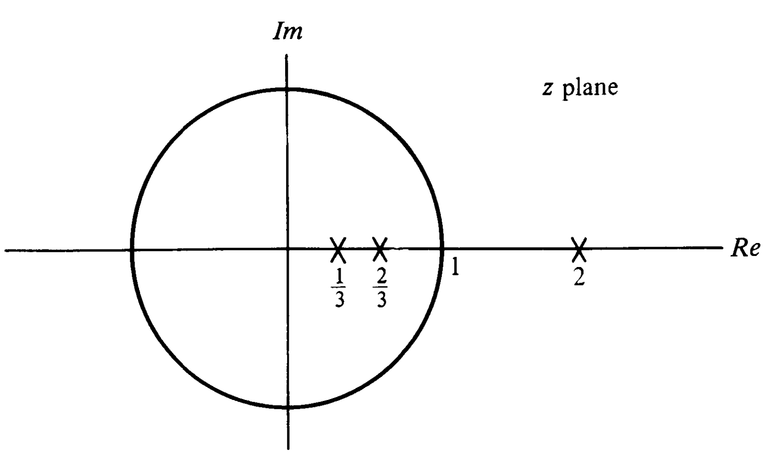

Figure P22.3

Determine what can be inferred about the associated region of convergence from each of the following statements.

(a) $x[n]$ is right-sided.

(b) The Fourier transform of $x[n]$ converges.

(c) The Fourier transform of $x[n]$ does not converge.

(d) $x[n]$ is left-sided.

P22.4

(a) Determine the z-transforms of the following two signals. Note that the z-transforms for both have the same algebraic expression and differ only in the ROC.

(i) $x_1[n] = (\frac{1}{2})^n u[n]$

(ii) $x_2[n] = -(\frac{1}{2})^n u[-n - 1]$

(b) Sketch the pole-zero plot and ROC for each signal in part (a).

(c) Repeat parts (a) and (b) for the following two signals:

(i) $x_3[n] = 2u[n]$

(ii) $x_4[n] = -(2)^n u[-n - 1]$

(d) For which of the four signals $x_1[n]$, $x_2[n]$, $x_3[n]$, and $x_4[n]$ in parts (a) and (c) does the Fourier transform converge?

P22.5

Consider the pole-zero plot of $H(z)$ given in Figure P22.5, where $H(a/2) = 1$.

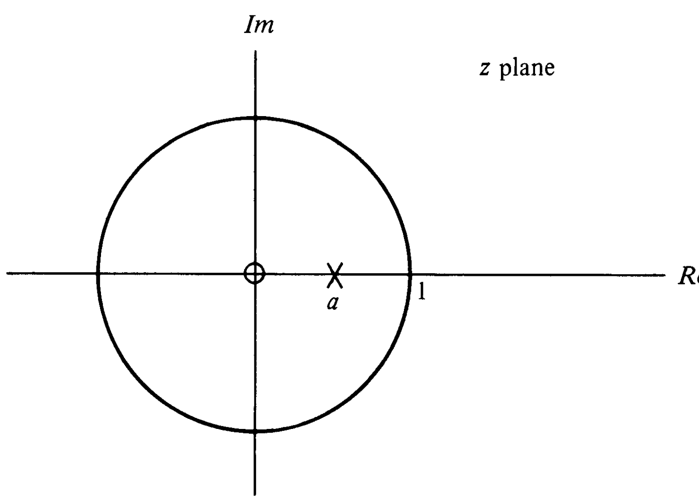

Figure P22.5

(a) Sketch $|H(e^{j\Omega})|$ as the number of zeros at $z = 0$ increases from 1 to 5.

(b) Does the number of zeros affect $\sphericalangle H(e^{j\Omega})$? If so, specifically in what way?

(c) Find the region of the z plane where $|H(z)| = 1$.

P22.6

Determine the z-transform (including the ROC) of the following sequences. Also sketch the pole-zero plots and indicate the ROC on your sketch.

(a) $(\frac{1}{3})^n u[n]$

(b) $\delta[n + 1]$

P22.7

For each of the following z-transforms determine the inverse z-transform.

(a) $X(z) = \dfrac{1}{1 + \frac{1}{2}z^{-1}}, \qquad |z| > \dfrac{1}{2}$

(b) $X(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}}, \qquad |z| > \dfrac{1}{2}$

(c) $X(z) = \dfrac{1 - az^{-1}}{z^{-1} - a}, \qquad |z| > \left|\dfrac{1}{a}\right|$

Optional Problems

P22.8

In this problem we study the relation between the z-transform, the Fourier transform, and the ROC.

(a) Consider the signal $x[n] = u[n]$. For which values of r does $r^{-n}x[n]$ have a converging Fourier transform?

(b) In the lecture, we discussed the relation between $X(z)$ and $\mathcal{F}\{r^{-n}x[n]\}$. For each of the following values of r, sketch where in the z plane $X(z)$ equals the Fourier transform of $r^{-n}x[n]$.

(i) $r = 1$

(ii) $r = \frac{1}{2}$

(iii) $r = 3$

(c) From your observations in parts (a) and (b), sketch the ROC of the z-transform of $u[n]$.

P22.9

(a) Suppose $X(z)$ on the circle $z = 2e^{j\Omega}$ is given by

$$X(2e^{j\Omega}) = \frac{1}{1 - \frac{1}{3}e^{-j\Omega}}$$

Using the relation $X(re^{j\Omega}) = \mathcal{F}\{r^{-n}x[n]\}$, find $2^{-n}x[n]$ and then $x[n]$, the inverse z-transform of $X(z)$.

(b) Find $x[n]$ from $X(z)$ below using partial fraction expansion, where $x[n]$ is known to be causal, i.e., $x[n] = 0$ for $n < 0$.

$$X(z) = \frac{3 + 2z^{-1}}{2 + 3z^{-1} + z^{-2}}$$

P22.10

A discrete-time system with the pole-zero pattern shown in Figure P22.10-1 is referred to as a first-order all-pass system because the magnitude of the frequency response is a constant, independent of frequency.

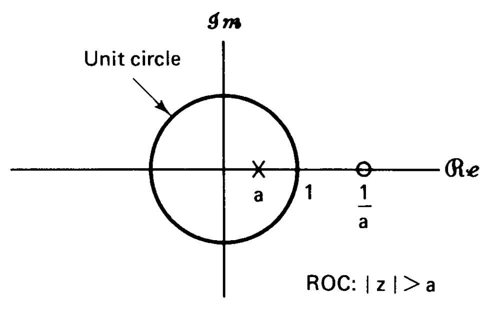

Figure P22.10-1

(a) Demonstrate algebraically that $|H(e^{j\Omega})|$ is constant.

(b) To demonstrate the same property geometrically, consider the vector diagram in Figure P22.10-2. Show that the length of v_2 is proportional to the length of v_1 independent of Ω by following these two steps:

(i) Express the length of v_1 using the law of cosines and the fact that it is one leg of a triangle for which the other two legs are the unit vector and a vector of length a.

(ii) In a manner similar to that in step (i), determine the length of v_2 and show that it is proportional in length to v_1 independent of Ω.

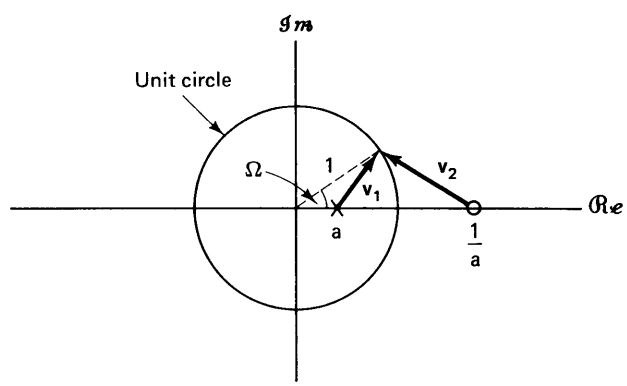

Figure P22.10-2

P22.11

Parts (a)–(e) (Figures P22.11-1 to P22.11-5) give pole-zero plots, and parts (i)–(iv) (Figures P22.11-6 to P22.11-9) give sketches of possible Fourier transform magnitudes. Assume that for all the pole-zero plots, the ROC includes the unit circle. For each pole-zero plot (a)–(e), specify which one *if any* of the sketches (i)–(iv) could represent the associated Fourier transform magnitude. More than one pole-zero plot may be associated with the same sketch.

(a)

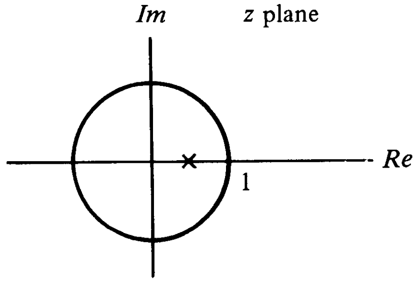

Figure P22.11-1

(b)

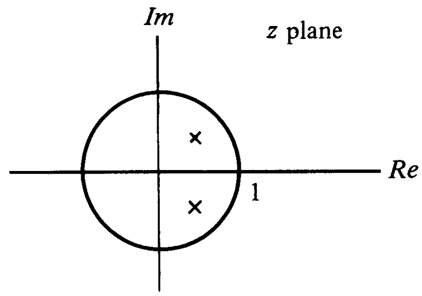

Figure P22.11-2

(c)

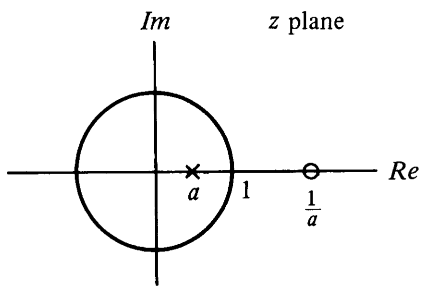

Figure P22.11-3

(d)

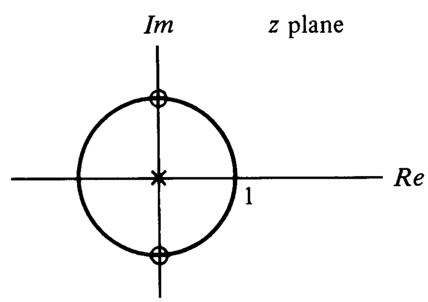

Figure P22.11-4

(e)

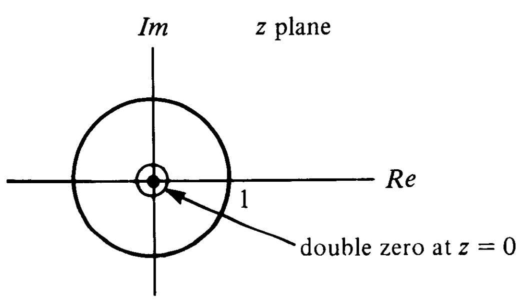

Figure P22.11-5

(i)

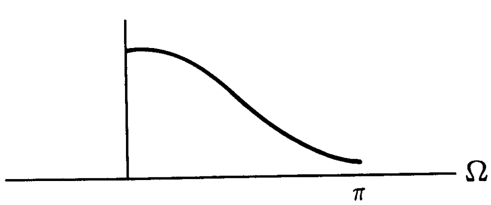

Figure P22.11-6

(ii)

Figure P22.11-7

(iii)

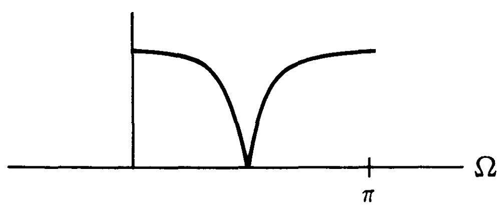

Figure P22.11-8

(iv)

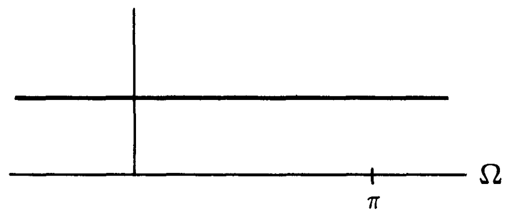

Figure P22.11-9

P22.12

Determine the z-transform for the following sequences. Express all sums in closed form. Sketch the pole-zero plot and indicate the ROC. Indicate whether the Fourier transform of the sequence exists.

(a) $(\frac{1}{2})^n\{u[n] - u[n - 10]\}$

(b) $(\frac{1}{2})^{|n|}$

(c) $7\left(\frac{1}{3}\right)^n \cos\left[\frac{2\pi n}{6} + \frac{\pi}{4}\right] u[n]$

(d) $x[n] = \begin{cases} 0, & n < 0 \\ 1, & 0 \le n \le 9 \\ 0, & 9 < n \end{cases}$

P22.13

Using the power-series expansion

$$\log(1 - w) = -\sum_{i=1}^{\infty} \frac{w^i}{i}, \qquad |w| < 1,$$

determine the inverse of the following z-transforms.

(a) $X(z) = \log(1 - 2z), \qquad |z| < \frac{1}{2}$

(b) $X(z) = \log(1 - \frac{1}{2}z^{-1}), \qquad |z| > \frac{1}{2}$