Some Results on Real-Part/Imaginary-Part and Magnitude-Phase Relations in Ambiguity Functions

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Summary—The uniqueness theorem for ambiguity functions states that if waveforms $u(t)$ and $v(t)$ have the same ambiguity function, *i.e.*, $\chi_u(\tau, \Delta) = \chi_v(\tau, \Delta)$, then $u(t)$ and $v(t)$ are identical except for a rotation, *i.e.*, $v(t) = e^{i\lambda}u(t)$, where λ is a real constant. Through the artifice of treating the even and odd parts of the waveforms, denoted $e(t)$ and $o(t)$, respectively, correlative results have been obtained for the real and imaginary parts of ambiguity functions. Thus, if $Re\ \{\chi_u(\tau, \Delta)\} = Re\ \{\chi_v(\tau, \Delta)\}$, then $e_v(t) = e^{i\lambda_e}e_u(t)$ and $o_v(t) = e^{i\lambda_o}o_u(t)$. From $Re\ \{\chi_u(\tau, \Delta)\}$, the waveform class $u(t) = e^{i\lambda}\,[e_u(t) + e^{ik}o_u(t)]$ may be constructed, but because of the aritrary rotation, e^{ik}, a unique χ_u-function is not determinable, in general. An important exception to this statement is the case when $\chi_u(\tau, \Delta)$ is real, and $Re\ \{\chi_u\} = \chi_u$ determines a unique waveform (within a rotation) and this waveform can only be even or odd.

If $Im\ \{\chi_u(\tau, \Delta)\} = Im\ \{\chi_v(\tau, \Delta)\}$ then $e_v(t) = ae^{i\gamma}e_u(t)$ and $o_v(t) = 1/ae^{i\gamma}o_u(t)$. If $Im\ \{\chi_u(\tau, \Delta)\}$ is given, *and* $u(t)$ is known to have unit energy, then within rotations of the form $e^{i\lambda}$, only two possible waveform choices are possible for $u(t)$. If it also is known which of $e_u(t)$ and $o_u(t)$ has the greater energy, the function $Im\ \{\chi_u(\tau, \Delta)\}$ uniquely determines $u(t)$ (within a rotation) and the complete χ_u-function.

The results on magnitude/phase relationships include a formula which enables one to compute the squared magnitude of an ambiguity function as an ordinary two-dimensional correlation function. Self-reciprocal two-dimensional Fourier transforms are demonstrated for the product of the squared-magnitude function and either of the first partial derivatives of the phase function.

I. Introduction

THIS PAPER presents a collection of results on real-part/imaginary-part and magnitude/phase relations in ambiguity functions. Its unifying purpose is that it is directed at the problem of how ambiguity functions might be used in the specifications and synthesis of waveforms, *i.e.*, continuous-wave coding, for communication systems. At the present time, the ambiguity function is an effective analytical tool, but it is not sufficiently well understood to serve as effectively as a synthesis tool. Gradually this difficulty is being overcome as the list of properties and theorems on ambiguity functions appearing in the literature increases [1]–[12]. One hopes that the present results can be added to this list.

The paper is composed of four sections. The remainder of this introduction contains definitions, properties, and theorems pertinent to the later developments. Section II is devoted to relations between the real and imaginary parts of ambiguity functions and their associated waveforms. Section III is devoted principally to the problem of the inter-relationship of magnitude and phase. The paper is concluded with a brief discussion.

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A. Definitions

Three equivalent formulas, giving the ambiguity function in symmetrical form are given in (1), (2), and (3).

$$\chi_{uv}(\tau, \Delta) = \begin{cases} \int u\left(t - \frac{\tau}{2}\right)\bar{v}\left(t + \frac{\tau}{2}\right)e^{-2\pi i\Delta t}\, dt & (1) \\ \int \bar{V}\left(f - \frac{\Delta}{2}\right)U\left(f + \frac{\Delta}{2}\right)e^{-2\pi i f\tau}\, df & (2) \\ e^{-\pi i\tau\Delta} \iint F_{uv}(t, f)e^{-2\pi i(\Delta t + \tau f)}\, dt\, df. & (3) \end{cases}$$

In these formulas, $u(t)$ and $v(t)$ are complex time functions, $U(f)$ and $V(f)$ are their Fourier transforms, and the function $F_{uv}(t, f)$ is given by

$$F_{uv}(t, f) = u(t)\bar{V}(f)e^{-2\pi i f t}. \tag{4}$$

τ and Δ are time- and frequency-shift variables, and all integrations (and elsewhere in this paper) are from $-\infty$ to ∞. If $u(t) \neq v(t)$, χ_{uv} is a time/frequency cross-correlation function (t/f ccf), and if $u(t) = v(t)$, it is a time/frequency autocorrelation function (t/f acf) which is denoted by a single subscript as in $\chi_u(\tau, \Delta)$.

The ambiguity function may be written either in terms of its real and imaginary parts

$$\chi_{uv}(\tau, \Delta) = R_{uv}(\tau, \Delta) + iI_{uv}(\tau, \Delta), \tag{5}$$

or its associated amplitude and phase functions

$$\chi_{uv}(\tau, \Delta) = |\chi_{uv}(\tau, \Delta)|\, e^{i\phi_{uv}(\tau, \Delta)}. \tag{6}$$

The squared-magnitude will appear quite frequently in the paper, and is, therefore, given a special symbol,

$$\psi_{uv}(\tau, \Delta) = |\chi_{uv}(\tau, \Delta)|^2. \tag{7}$$

B. Properties and Theorems of Interest to This Paper

Most of the following properties and theorems are given in the reference, and all may be obtained through straightforward manipulations using the definitions.

1) *Symmetry Properties*: Ambiguity functions have the following symmetry properties:

For t/f ccfs

$$\chi_{uv}(\tau, \Delta) = \bar{\chi}_{vu}(-\tau, -\Delta) \tag{8}$$

and

$$\psi_{uv}(\tau, \Delta) = \psi_{vu}(-\tau, -\Delta). \tag{9}$$

For t/f acfs

$$\chi_u(\tau, \Delta) = \bar{\chi}_u(-\tau, -\Delta) \tag{10}$$

and

$$\psi_u(\tau, \Delta) = \psi_u(-\tau, -\Delta). \tag{11}$$

Eq. (10) states that

$$|\chi_u(\tau, \Delta)|\, e^{i\phi_u(\tau,\Delta)} = |\chi_u(-\tau, -\Delta)|\, e^{-i\phi_u(-\tau,-\Delta)},$$

so that in view of (11), the phase function satisfies

$$\phi_u(\tau, \Delta) = -\phi_u(-\tau, -\Delta). \tag{12}$$

Thus, ϕ_u is an odd function, and may only contain a part which is even in τ, odd in Δ, and a part which is odd in τ, even in Δ, *i.e.*, no even terms (even in both variables or odd in both variables) are present in ϕ_u. It follows that the *relative phase* between two ambiguity functions having the same magnitude, a matter discussed incorrectly by the author in an earlier paper, must also be an odd function [10].[1]

The corresponding statement for the phase of t/f ccfs is

$$\phi_{uv}(\tau, \Delta) = -\phi_{vu}(-\tau, -\Delta). \tag{13}$$

2) *Uniqueness Properties*: Except for trivial factors, ambiguity functions uniquely determine the associated time functions. The uniqueness of t/f acfs is stated in a theorem first published by Wilcox [3].

Theorem 1: If $\chi_v(\tau, \Delta) = \chi_u(\tau, \Delta)$, then $v(t)$ is identical with $u(t)$ except for a rotation, *i.e.*, $v(t) = e^{i\lambda}u(t)$, where λ is a real constant.

The corresponding theorem for t/f ccfs follows.

Theorem 2: If for waveforms of unit energy, $\chi_{\mu\nu}(\tau, \Delta) = \chi_{uv}(\tau, \Delta)$, then $\mu(t) = e^{i\lambda}u(t)$ and $\nu(t) = e^{i\lambda}v(t)$, where λ is a real constant.

A proof of Theorem 2, which is repeated from Stutt [10], depends on a necessary and sufficient condition on χ_u functions given by Wilcox [3] and Siebert [4]. This condition is given as Theorem 3, and is generalized to include χ_{uv} functions.

Theorem 3: In order to be the t/f ccf, $\chi_{uv}(\tau, \Delta)$, the square integrable function $\zeta(\tau, \Delta)$ must satisfy the following factorization condition:

$$\int \zeta(\eta - \xi, \Delta) e^{\pi i \Delta(\eta+\xi)}\, d\Delta = u(\xi)\bar{v}(\eta).$$

The condition in Theorem 3 is essentially the inverse of (1) with $t - \tau/2 = \xi$, $t + \tau/2 = \eta$.

Thus, if $\chi_{\mu\nu}(\tau, \Delta) = \chi_{uv}(\tau, \Delta)$, then it must be true that

$$\mu(\xi)\bar{\nu}(\eta) = u(\xi)\bar{v}(\eta).$$

This equality holds if $\mu(\xi) = ku(\xi)$ and $\bar{\nu}(\eta) = 1/k\bar{v}(\eta)$, but since by the energy constraint, $E_\mu = E_u = E_\nu = E_v = 1$, then $|k| = |1/k| = 1$, *i.e.*, $k = e^{i\lambda}$.

[1] The phase symmetry relation (12) became evident to both C. H. Wilcox and the author following a discussion of the symmetry property of relative phase. It turns out to be much easier to deduce the symmetry relation of relative phase from that of total phase, than to show it directly.

3) *Two-Dimensional Fourier Transform Properties*: Two-dimensional Fourier transforms of ambiguity functions exhibit some of their interesting properties. The transform of a t/f ccf between $u(t)$ and $v(t)$ is another t/f ccf between $u(t)$ and the time inverse of $v(t)$

$$\iint \chi_{uv}(\sqrt{2}\tau, \sqrt{2}\Delta) e^{2\pi i(\tau y - \Delta x)}\, d\tau\, d\Delta = \chi_{u\mathbf{v}}(\sqrt{2}x, \sqrt{2}y) \tag{14}$$

where the underscore denotes the time inverse, $\mathbf{v}(t) \equiv v(-t)$ and x and y are time-shift and frequency-shift variables as are τ and Δ. For a t/f acf the relationship is

$$\iint \chi_{u}(\sqrt{2}\tau, \sqrt{2}\Delta) e^{2\pi i(\tau y - \Delta x)}\, d\tau\, d\Delta = \chi_{u\mathbf{u}}(\sqrt{2}x, \sqrt{2}y) \tag{15}$$

and the function $\chi_{u\mathbf{u}}(\sqrt{2}x, \sqrt{2}y)$, which is essentially the Wigner distribution in the Statistical Theory of Quantum Mechanics, is a real function, as an evaluation of its conjugate readily demonstrates [9].

The following two-dimensional transform involving t/f ccfs between u_1, u_2, u_3, and u_4 is a very useful one, and was first published by Sussman [12],

$$\iint \chi_{12}(\tau, \Delta)\bar{\chi}_{34}(\tau, \Delta) e^{2\pi i(\tau y - \Delta x)}\, d\tau\, d\Delta = \chi_{13}(x, y)\bar{\chi}_{24}(x, y). \tag{16}$$

In the form given here, only the inner indices are interchanged in the transformation. If $u_1 = u_3$ and $u_2 = u_4$, the transform gives a relationship between the magnitude of a t/f ccf and the associated t/f acfs [10],

$$\iint \psi_{12}(\tau, \Delta) e^{2\pi i(\tau y - \Delta x)}\, d\tau\, d\Delta = \chi_1(x, y)\bar{\chi}_2(x, y). \tag{17}$$

If $u_1 = u_2 = u_3 = u_4 = u$, the self-reciprocal property of the ψ_u functions is obtained [4],

$$\iint \psi_u(\tau, \Delta) e^{2\pi i(\tau y - \Delta x)}\, d\tau\, d\Delta = \psi_u(x, y). \tag{18}$$

If one chooses $u_1 = u$ and $u_2 = \mathbf{u}$, then (17) yields

$$\iint \psi_{u\mathbf{u}}(\tau, \Delta) e^{2\pi i(\tau y - \Delta x)}\, d\tau\, d\Delta = \chi_u(x, y)\bar{\chi}_{\mathbf{u}}(x, y), \tag{19}$$

but $\chi_{u\mathbf{u}}$ is real so that $\psi_{u\mathbf{u}} = \chi_{u\mathbf{u}}^2$, and $\bar{\chi}_{\mathbf{u}}(x, y)$ is readily seen from (1) to be $\chi_u(x, y)$. Thus

$$\iint \chi_{u\mathbf{u}}^2(\tau, \Delta) e^{2\pi i(\tau y - \Delta x)}\, d\tau\, d\Delta = \chi_u^2(x, y). \tag{20}$$

A comparison of (15) and (20) shows the interesting fact that not only do χ_u and $\chi_{u\mathbf{u}}$ form a transform pair, but their squares χ_u^2 and $\chi_{u\mathbf{u}}^2$ also form a pair.

II. Real-Part/Imaginary-Part Relations

A. General Expressions for R_u and I_u

In view of the symmetry property for χ_u in (10) the real and imaginary parts of χ_u in (5) are given by

$$R_u(\tau, \Delta) = \tfrac{1}{2}[\chi_u(\tau, \Delta) + \chi_u(-\tau, -\Delta)] \tag{21}$$

and

$$I_u(\tau, \Delta) = \frac{1}{2i}[\chi_u(\tau, \Delta) - \chi_u(-\tau, -\Delta)]. \tag{22}$$

Substituting for χ_u from (3), one finds that R_u and I_u are the $e^{-\pi i\tau\Delta}$—multiplied cosine and sine transforms of $F(t, f)$, a somewhat surprising result inasmuch as both $F(t, f)$ and $e^{-\pi i\tau\Delta}$ are complex.

$$R_u(\tau, \Delta) = e^{-\pi i\tau\Delta} \iint F(t, f) \cos 2\pi(\Delta t + \tau f)\, dt\, df \tag{23}$$

$$I_u(\tau, \Delta) = -e^{-\pi i\tau\Delta} \iint F(t, f) \sin 2\pi(\Delta t + \tau f)\, dt\, df. \tag{24}$$

An exactly parallel pair of equations is obtained for t/f ccfs; however, the sum $\chi_{uv}(\tau, \Delta) + \chi_{vu}(\tau, \Delta)$ must be used, whereupon an application of the symmetry rules gives

$$R_{uv}(\tau, \Delta) + R_{vu}(\tau, \Delta)$$
$$= e^{-\pi i\tau\Delta} \iint [F_{uv}(t, f) + F_{vu}(t, f)] \cos 2\pi(\Delta t + \tau f)\, dt\, df \tag{25}$$

and

$$I_{uv}(\tau\Delta) + I_{vu}(\tau\Delta)$$
$$= -e^{-\pi i\tau\Delta} \iint [F_{uv}(t, f) + F_{vu}(t, f)] \sin 2\pi(\Delta t + \tau f)\, dt\, df. \tag{26}$$

B. Expressions for R_u and I_u in Terms of Even and Odd Time Functions

In general, a time function may be decomposed into even and odd parts, both of which may be complex,

$$u(t) = e(t) + o(t). \tag{27}$$

The t/f acf of $u(t)$ may then be decomposed into four parts, namely,

$$\chi_u(\tau, \Delta) = \chi_e(\tau, \Delta) + \chi_0(\tau, \Delta) + \chi_{e0}(\tau, \Delta) + \chi_{0e}(\tau, \Delta) \tag{28}$$

where χ_e and χ_0 are the t/f acfs of $e(t)$ and $o(t)$ and χ_{e0} and χ_{0e} are their t/f ccfs. Because of the even and odd natures of $e(t)$ and $o(t)$, calculations of the conjugates, using (1), of the four terms in (28) show that both χ_e and χ_0 are real and the sum, $\chi_{e0} + \chi_{0e}$, is purely imaginary, hence,

$$R_u(\tau, \Delta) = \chi_e(\tau, \Delta) + \chi_0(\tau, \Delta) \tag{29a}$$

and

$$I_u(\tau, \Delta) = -i[\chi_{e0}(\tau, \Delta) + \chi_{0e}(\tau, \Delta)]. \tag{29b}$$

The transform function $\chi_{uu}(x, y)$ in (15) also may be decomposed into the same parts with appropriate changes in sign

$$\chi_{uu}(x, y) = \chi_e(x, y) - \chi_0(x, y) - \chi_{e0}(x, y) + \chi_{0e}(x, y) \tag{30}$$

and there is, of course, a term-by-term transform correspondence between the terms in $\chi_u(\tau, \Delta)$ and those in $\chi_{uu}(x, y)$.

It is remarked above that χ_e and χ_0 are both real. The reverse statement is also true; these statements may be combined into a theorem.

Theorem 4: A t/f acf is real if and only if the associated time function is even or odd.[2]

It remains only to prove the sufficiency of this theorem. If $\chi_u(\tau, \Delta)$ is real, then from (29b),

$$\chi_{e0} + \chi_{0e} = 0,$$

which implies from Theorem 3 that

$$e(x)\bar{o}(y) + o(x)\bar{e}(y) = 0.$$

If $e(t)$ and $o(t)$ are not disjoint, then there is a y for which both $\bar{o}(y)$ and $\bar{e}(y)$ are different from zero, but this makes it necessary for $e(x)$ and $o(x)$ to be proportional, which is impossible in view of their even and odd natures. Thus if they are not disjoint, either $e(t)$ or $o(t)$ must be zero. If $e(t)$ and $o(t)$ are disjoint [$e(t)\ o(t) = 0$ all t] but neither is identically zero, then there is a y for which $\bar{o}(y) \neq 0$ and $e(y) = 0$. But the above relation would then require $e(x) = 0$ for all x, which is contrary to the hypothesis. Likewise $o(x) = 0$ for all x at some y for which $e(y) \neq 0$ and $\bar{o}(y) = 0$. Clearly, then, if $\chi_{e0} + \chi_{0e} = 0$, one or the other of $e(t)$ and $o(t)$ must be zero, *i.e.* $u(t)$ is either even or odd.

C. Deductions from Known R_u and I_u Functions

The uniqueness theorem for ambiguity functions states that except for a rotation a χ_u-function uniquely determines the associated waveform, $u(t)$. In this section, the extent to which the real part $R_u(\tau, \Delta)$ and the imaginary part $I_u(\tau, \Delta)$ affect this uniqueness will be considered.

First, it will be noted that $R_u(\tau, \Delta)$ determines both the even and odd parts of the waveform within arbitrary rotations. A straightforward application of Theorem 3 to the two terms in (29a) gives

$$\int R_u(y - x, \Delta) e^{\pi i\Delta(y+x)}\, d\Delta$$
$$= \int [\chi_e(y - x, \Delta) + \chi_0(y - x, \Delta)] e^{\pi i\Delta(y+x)}\, d\Delta$$
$$= e(x)\bar{e}(y) + o(x)\bar{o}(y), \tag{31}$$

[2] The question of sufficiency in this theorem arose and was proved in a discussion with C. H. Wilcox.

whereupon by virtue of the even and odd characters of the waveforms, it is readily seen that

$$e(x)\bar{e}(y) = \frac{1}{2}\int [R_u(y - x, \Delta)e^{\pi i \Delta x} + R_u(y + x, \Delta)e^{-\pi i \Delta x}]e^{\pi i \Delta y}\, d\Delta \quad (32)$$

and

$$o(x)\bar{o}(y) = \frac{1}{2}\int [R_u(y - x, \Delta)e^{\pi i \Delta x} - R_u(y + x, \Delta)e^{-\pi i \Delta x}]e^{\pi i \Delta y}\, d\Delta. \quad (33)$$

Thus $e_u(t) = e^{i\lambda_e}e(t)$ and $o_u(t) = e^{i\lambda_0}o(t)$ are the possible forms of the components of $u(t) = e_u(t) + o_u(t)$, where λ_e and λ_0 are arbitrary real constants. Clearly, if there is another waveform $v(t)$ for which $R_v(\tau, \Delta) = R_u(\tau, \Delta)$, then the even and odd components of $v(t)$ can differ only by arbitrary rotations from those of $u(t)$,

$$e_v(t) = e^{i\alpha}e_u(t), \qquad o_v(t) = e^{i\beta}o_u(t). \quad (34)$$

By writing the arbitrary constant in the form $\lambda_0 = \lambda_e + \kappa$, where κ is also arbitrary, the general form of $u(t)$ reproducing $R_u(\tau, \Delta)$ and only $R_u(\tau, \Delta)$, is

$$u(t) = e^{i\lambda_e}[e(t) + e^{i\kappa}o(t)]. \quad (35)$$

This waveform, however, does not produce a unique I_u-function, *i.e.*,

$$I_u(\tau, \Delta) = -i[e^{-i\kappa}\chi_{e0}(\tau, \Delta) + e^{i\kappa}\chi_{0e}(\tau, \Delta)] = 2[I_{e0}(\tau, \Delta)\cos\kappa - R_{e0}(\tau, \Delta)\sin\kappa], \quad (36)$$

which function is clearly dependent on the arbitrary choice of κ. Thus $R_u(\tau, \Delta)$ does not determine a unique χ_u-function, except, of course, under the condition of Theorem 4 in which case $\chi_u(\tau, \Delta)$ is pure real and $u(t)$ is either even or odd.

The situation is somewhat different when the imaginary part of the ambiguity function is given instead of the real part. In fact, when certain additional energy constraints are placed on the waveform, the I_u-function uniquely determines the complete χ_u- function.

If $I_u(\tau, \Delta)$ is known, then (29b) and Theorem 3 show that

$$i\int I_u(y - x, \Delta)e^{\pi i \Delta(y+x)}\, d\Delta = e(x)\bar{o}(y) + o(x)\bar{e}(y). \quad (37)$$

Thence, by the even and odd natures of $e(t)$ and $o(t)$,

$$e(x)\bar{o}(y) = \frac{i}{2}\int [I_u(y - x, \Delta)e^{\pi i \Delta x} + I_u(y + x, \Delta)e^{-\pi i \Delta x}]e^{\pi i \Delta y}\, d\Delta. \quad (38)$$

This equation specifies the forms of the even and odd parts of the waveform, but not their relative energy. Define $m(t)$ to be an even function of the exact form of $e(t)$ in (38), but scaled to have unit energy, $E_m = 1$, *i.e.*, $m(t) = 1/\sqrt{E_e}e(t)$; likewise define $n(t)$ to be an odd function of the exact form of $o(t)$ in (38), but scaled to have unit energy, $E_m = 1$, *i.e.*, $n(t) = 1/\sqrt{E_0}e(t)$. Thus,

$$e(x)\bar{o}(y) = \sqrt{E_e E_0}\, m(x)\bar{n}(y). \quad (39)$$

Without further constraint on the energy of the waveform, the even and odd parts of $u(t)$, may be written as

$$e_u(t) = a\sqrt{E_e}e^{i\gamma}m(t) \quad (40a)$$

$$o_u(t) = \frac{1}{a}\sqrt{E_0}e^{i\gamma}n(t). \quad (40b)$$

Clearly, any other waveform $v(t)$ for which $I_v(\tau, \Delta) = I_u(\tau, \Delta)$ has even and odd components which can differ from those of $u(t)$ only in the real constants a and γ.

If, however, the total waveform energy is known, *i.e.*, $E_u = E_e + E_0 = 1$, then energy assignments can be made. Because the I_u-function is known, the product of the energies of $e(t)$ and $o(t)$ is calculable,

$$P \equiv 2E_eE_0 = \iint I_u^2(\tau, \Delta)\, d\tau\, d\Delta. \quad (41)$$

This relationship together with the sum relationship above enables one to find the following alternative values for E_e and E_0:

$$E_e = \tfrac{1}{2}[1 \pm \sqrt{1 - 2P}] \quad (42)$$

$$E_0 = \tfrac{1}{2}[1 \mp \sqrt{1 - 2P}], \quad (43)$$

whereupon, the only possible choices for $u(t)$ are seen to be

$$u_1(t) = \frac{e^{i\gamma_1}}{\sqrt{2}}\{[1 + \sqrt{1 - 2P}]^{1/2}m(t) + [1 - \sqrt{1 - 2P}]^{1/2}n(t)\} \quad (44)$$

and

$$u_2(t) = \frac{e^{i\gamma_2}}{\sqrt{2}}\{[1 - \sqrt{1 - 2P}]^{1/2}m(t) + [1 + \sqrt{1 - 2P}]^{1/2}n(t)\}. \quad (45)$$

Thus, except for arbitrary rotations, the specification of the I_u-function and the waveform energy gives two alternative choices for $u(t)$. Note that the ambiguity of two waveforms is removed if in addition, it is also known which of the even and odd parts of the waveform has the larger energy.

III. Phase/Amplitude Relationships

A. ψ_u Expressed as an Ordinary Two-Dimensional Correlation Function

From the double integral form for χ_u, in (3), the squared magnitude function is given by

$$\psi_u(\tau, \Delta) = \iiiint F_u(t, f)\bar{F}_u(s, g) \cdot e^{-2\pi i[\tau(f-g)+\Delta(t-s)]}\, dt\, df\, ds\, dg, \quad (46)$$

or with $f - g = y$ and $t - s = -x$,

$$\psi_u(\tau, \Delta) = \iiiint F_u(t, f)\bar{F}_u(t + x, f - y) \cdot e^{-2\pi i(\tau y - \Delta x)}\, dt\, df\, dx\, dy. \tag{47}$$

Thus, there is the transform relation

$$\iint \psi_u(\tau, \Delta)e^{2\pi i(\tau y - \Delta x)}\, d\tau\, d\Delta = \iint F_u(t, f)\bar{F}_u(t + x, f - y)\, dt\, df. \tag{48}$$

But in view of the self-reciprocal transform property of the ψ_u functions given in (18), it follows that

$$\psi_u(x, y) = \iint F_u(t, f)\bar{F}_u(t + x, f - y)\, dt\, df. \tag{49}$$

Thus the squared magnitude of an ambiguity function may be generated as the ordinary two-dimensional autocorrelation of the function $F_u(t, f) = u(t)\bar{U}(f)e^{-2\pi i f t}$.

B. Self-Reciprocal Transforms Involving ψ, $\partial\phi/\partial\tau$, and $\partial\phi/\partial\Delta$

It will be convenient in this development to use the function

$$\Omega_u(\tau, \Delta) \equiv e^{\pi i \tau \Delta}\chi_u(\tau, \Delta) = \iint F_u(t, f)e^{-2\pi i(\tau f + \Delta t)}\, dt\, df, \tag{50}$$

i.e., Ω_u is a two-dimensional transform of F_u. Clearly, $|\Omega_u| = |\chi_u|$ and $\psi_u = |\Omega_u|^2$. If the phase of Ω_u is denoted by θ_u, then

$$\theta_u(\tau, \Delta) = \phi_u(\tau, \Delta) + \pi\tau\Delta. \tag{51}$$

Now,

$$e^{2i\theta_u(\tau,\Delta)} = \frac{\Omega_u(\tau, \Delta)}{\bar{\Omega}_u(\tau, \Delta)},$$

from which the partial derivative of θ_u with respect to τ is found to be

$$2i\frac{\partial\theta_u}{\partial\tau} = \frac{1}{\Omega_u}\frac{\partial\Omega_u}{\partial\tau} - \frac{1}{\bar{\Omega}_u}\frac{\partial\bar{\Omega}_u}{\partial\tau},$$

whereupon

$$2i\psi_u\frac{\partial\theta_u}{\partial\tau} = M(\tau, \Delta), \tag{52}$$

in which

$$M(\tau, \Delta) = \bar{\Omega}_u\frac{\partial\Omega_u}{\partial\tau} - \Omega_u\frac{\partial\bar{\Omega}_u}{\partial\tau}. \tag{53}$$

The function M can be put into a more useful form by means of (50),

$$\frac{\partial\Omega_u(\tau, \Delta)}{\partial\tau} = -2\pi i\iint fu(t)\bar{U}(f)e^{-2\pi i f t}e^{-2\pi i(\tau f + \Delta t)}\, dt\, df. \tag{54}$$

If a spectrum function $U_1(f)$ with transform $u_1(t)$ is defined to be

$$U_1(f) = 2\pi i f U(f), \tag{55}$$

then it is seen that

$$\frac{\partial\Omega_u(\tau, \Delta)}{\partial\tau} = \Omega_{uu_1}(\tau, \Delta). \tag{56}$$

M can then be expressed in the form

$$M(\tau, \Delta) = \Omega_{uu_1}(\tau, \Delta)\bar{\Omega}_u(\tau, \Delta) - \Omega_u(\tau, \Delta)\bar{\Omega}_{uu_1}(\tau, \Delta). \tag{57}$$

The next step is to take the two-dimensional transform of both sides of (52),

$$2i\iint \psi_u(\tau, \Delta)\frac{\partial\theta_u(\tau, \Delta)}{\partial\tau}e^{2\pi i(\tau y - \Delta x)}\, d\tau\, d\Delta = \iint M(\tau, \Delta)e^{2\pi i(\tau y - \Delta x)}\, d\tau\, d\Delta. \tag{58}$$

It is noted that because the $e^{\pi i \tau \Delta}$ factors cancel out in conjugation, there is the equality

$$\chi_{12}(\tau, \Delta)\bar{\chi}_{34}(\tau, \Delta) = \Omega_{12}(\tau, \Delta)\bar{\Omega}_{34}(\tau, \Delta) \tag{59}$$

and, therefore, the Ω functions also satisfy the transform relation given in (16). With this in mind, (58) is seen to give

$$2i\iint \psi_u(\tau, \Delta)\frac{\partial\theta_u(\tau, \Delta)}{\partial\tau}e^{2\pi i(\tau y - \Delta x)}\, d\tau\, d\Delta = \Omega_u(x, y)\bar{\Omega}_{u_1u}(x, y) - \Omega_u(x, y)\bar{\Omega}_{uu_1}(x, y), \tag{60}$$

where the form of the second term on the right is invariant in the transformation. It will be necessary to find an alternate expression for $\bar{\Omega}_{u_1u}(x, y)$, and this may be done by evaluating $\bar{\Omega}_{uu_1}(x, y)$, which is

$$\begin{aligned}\bar{\Omega}_{uu_1}(x, y) &= \iint \bar{F}_{uu_1}(t, f)e^{2\pi i(xf + yt)}\, dt\, df\\ &= \iint \bar{u}(t)[2\pi i f U(f)]e^{2\pi i f t}e^{2\pi i(xf + yt)}\, dt\, df\\ &= 2\pi i\iiiint f\bar{U}(g)e^{-2\pi i g t}u(s)e^{-2\pi i f s}e^{2\pi i f t}e^{2\pi i(xf + yt)}\, dt\, df\, ds\, dg\\ &= -(-2\pi i)e^{-2\pi i x y}\iint (g - y)u(s)\bar{U}(g)e^{-2\pi i g s}e^{2\pi i(xg + ys)}\, ds\, dg\\ &= -e^{-2\pi i x y}\Omega_{uu_1}(-x, -y) - 2\pi i y e^{-2\pi i x y}\Omega_u(-x, -y).\end{aligned} \tag{61}$$

The symmetry rule for the Ω-functions is readily found from the rule for χ-functions, (8), *i.e.*,

$$e^{-\pi i x y}\Omega_{uv}(x, y) = e^{\pi i x y}\bar{\Omega}_{vu}(-x, -y)$$

or

$$e^{-2\pi i x y}\Omega_{vu}(-x, -y) = \bar{\Omega}_{uv}(x, y). \tag{62}$$

This rule shows that

$$e^{-2\pi i x y}\Omega_{uu_1}(-x, -y) = \bar{\Omega}_{u_1u}(x, y) \tag{63}$$

and

$$e^{-2\pi i x y}\Omega_u(-x, -y) = \bar{\Omega}_u(x, y), \tag{64}$$

whereupon (61) becomes

$$\bar{\Omega}_{uu_1}(x, y) = -\bar{\Omega}_{u_1u}(x, y) - 2\pi iy\bar{\Omega}_u(x, y). \tag{65}$$

This equation may be solved for $\bar{\Omega}_{u_1u}$, which, when inserted in (60), gives

$$2i \iint \psi_u(\tau, \Delta) \frac{\partial \theta_u(\tau, \Delta)}{\partial \tau} e^{2\pi i(\tau y - \Delta x)} \, d\tau \, d\Delta = -2\pi iy \, |\Omega_u(x, y)|^2 - 2\Omega_u(x, y)\bar{\Omega}_{uu_1}(x, y). \tag{66}$$

On the right hand side, $|\Omega_u|^2$ is identical with ψ_u,

$$\Omega_u(x, y) = |\Omega_u(x, y)| \, e^{i\theta_u(x,y)}$$

and by (56),

$$\bar{\Omega}_{uu_1}(x, y) = \frac{\partial \bar{\Omega}_u(x, y)}{dx} = \frac{\partial \, |\Omega_u(x, y)|}{\partial x} e^{-i\theta(x,y)} - i \, |\Omega_u(x, y)| \, e^{-i\theta(x,y)} \frac{\partial \theta(x, y)}{\partial x}$$

hence

$$2i \iint \psi_u(\tau, \Delta) \frac{\partial \theta_u(\tau, \Delta)}{\partial \tau} e^{2\pi i,\tau y - \Delta x)} \, d\tau \, d\Delta = -2\pi iy\psi_u(x, y) - \frac{\partial \psi_u(x, y)}{\partial x} + 2i\psi_u(x, y) \frac{\partial \theta_u(x, y)}{\partial x} \tag{67}$$

where

$$\frac{\partial \psi_u(x, y)}{\partial x} = 2 \, |\Omega_u(x, y)| \, \frac{\partial \, |\Omega_u(x, y)|}{\partial x}.$$

It is appropriate to express (67) in terms of ϕ instead of θ, which may be done with the aid of (51), thus,

$$\frac{\partial \theta_u(x, y)}{\partial x} = \frac{\partial \phi_u(x, y)}{\partial x} + \pi y \tag{68}$$

and so (67) becomes

$$2i \iint \psi_u(\tau, \Delta)\left[\pi\Delta + \frac{\partial \phi_u(\tau, \Delta)}{\partial \tau}\right] e^{2\pi i(\tau y - \Delta x)} \, d\tau \, d\Delta = -\frac{\partial \psi_u(x, y)}{\partial x} + 2i\psi_u(x, y) \frac{\partial \phi_u(x, y)}{\partial x}. \tag{69}$$

Now it is a property of Fourier transforms, including the present two-dimensional transform, that the even-odd character of functions is preserved in the transformation. On the left, $\psi_u(\tau, \Delta)$ obeying symmetry rule (11), has only even-even and odd-odd parts, therefore, $\Delta\psi_u(\tau, \Delta)$ has even-odd and odd-even parts; $\phi_u(\tau, \Delta)$ has even-odd and odd-even parts by (12), hence $\partial\phi_u(\tau, \Delta)/\partial\tau$ and the product $\psi_u(\tau, \Delta) \; \partial\phi_u(\tau, \Delta)/\partial\tau$ have only odd-odd and even-even parts. On the right $\partial\psi_u(x, y)/\partial x$ has odd-even and even-odd parts, and the product $\psi_u(x, y) \; \partial\phi_u(x, y)/\partial x$, again, has odd-odd and even-even parts. Consequently, upon associating terms of the same even-odd character, (69) breaks into the two equations

$$-2\pi i \iint \Delta\psi_u(\tau, \Delta) e^{2\pi i(\tau y - \Delta x)} \, d\tau \, d\Delta = \frac{\partial \psi_u(x, y)}{\partial x} \tag{70}$$

and

$$\iint \psi_u(\tau, \Delta) \frac{\partial \phi_u(\tau, \Delta)}{\partial \tau} e^{2\pi i(\tau y - \Delta x)} \, d\tau \, d\Delta = \psi_u(x, y) \frac{\partial \phi_u(x, y)}{\partial x}. \tag{71}$$

The first of these two equations contributes nothing new since it may be obtained by differentiation from the self-reciprocal property of the ψ_u-functions in (18). In fact (18) could have been used to eliminate the two terms of (70) from (69), without resort to the even-odd preserving property of Fourier transforms. The second equation is new, however, and states that the product $\psi_u(\tau, \Delta) \; \partial\phi_u(\tau, \Delta)/\partial\tau$ is self-reciprocal under Fourier transformation.

Similar results would have been obtained using the partial derivative of the phase function with respect to the frequency-shift variable Δ. In this case a function

$$N(\tau, \Delta) = \bar{\Omega}_u(\tau, \Delta) \frac{\partial \Omega_u(\tau, \Delta)}{\partial \Delta} - \Omega_u(\tau, \Delta) \frac{\partial \bar{\Omega}_u(\tau, \Delta)}{\partial \Delta} \tag{72}$$

is obtained, and with aid of the artifice

$$u_2(t) \equiv -2\pi itu(t). \tag{73}$$

N can be put into a form which has a readily recognizable transform. The end result of these parallel manipulations is the self-reciprocal transformation

$$\iint \psi_u(\tau, \Delta) \frac{\partial \phi_u(\tau, \Delta)}{\partial \Delta} e^{2\pi i(\tau y - \Delta x)} \, d\tau \, d\Delta = \psi_u(x, y) \frac{\partial \phi_u(x, y)}{\partial y}. \tag{74}$$

IV. Discussion

The representation of a waveform by its even and odd parts has exhibited some of the interesting properties of ambiguity functions. It was seen that the real part and imaginary part of ambiguity functions are readily identified in terms of the t/f correlation functions of these even and odd parts, and that the transform function $\chi_{uu}(x, y)$ may also be written immediately from these same t/f correlation functions. It is interesting that from the real part of an ambiguity function, the form of the even and odd components of the associated waveform and their respective energies may be obtained. The real part, however, does not specify the relative rotation of these two components which is needed to construct a waveform having a unique ambiguity function, *i.e.*, there is an infinite number of waveforms which give ambiguity functions having the same real part. If the ambiguity function itself, is real, however, there is only one nontrivial waveform which may be associated with it, and it must be either even or odd.

The imaginary part of an ambiguity function determines the form of the even and odd parts of the waveform, but not their respective energies. However, if the total waveform energy is also known, then only two energy choices are possible for these components and it turns out that with this additional information, the imaginary part of an ambiguity function then determines two and only two complete ambiguity functions. If there is the further information as to which of the even and odd components of the waveform has the greater energy, then this imaginary part determines a unique ambiguity function.

The expression (49) for the squared magnitude of a t/f acf may offer an alternate, if not more convenient, means of calculating this magnitude function from a knowledge of the waveform. It apparently sheds no light on the standing question as to how to test for a magnitude function.

That $\psi_u \, \partial\phi_u/\partial\tau$ and $\psi_u \, \partial\phi_u/\partial\Delta$ are self-reciprocal functions in a two-dimensional Fourier transformation may be the most significant result of the paper. To the author's knowledge, these are the first expressions showing a direct tie between the magnitude and phase of an ambiguity function. The relationship, of course, is not unique, *e.g.*, if ϕ_1, which gives rise to waveform $u_1(t)$ satisfies the two transform relations (71) and (74), then surely $\phi_2 = \phi_1 + a\tau + b\Delta$ and $\phi_3 = -\phi_1$ do, too. In the first case, the linear phase terms give rise to a waveform $u_2(t)$ which is a time-frequency translate of $u_1(t)$, and in the second, the t/f acf is conjugated and the associated waveform $u_3(t)$ is the same as $u_1(t)$ except that the sign of its odd part is reversed. More general solutions to these self-reciprocal relations is beyond the scope of the present paper, and is the subject of further study.

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