DFT Beamforming Algorithms for Space-Time-Frequency Applications

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ABSTRACT

This work presents modified variants, in a recursive format, of the Kahaner's additive fast Fourier transform (FFT) algorithm. The variants are presented in Kronecker products algebra language. The language serves as a tool for the analysis, design, modification and implementation of the FFT variants on re-configurable field programmable gate array (FPGA) computational structures. The target for these computational structures are discrete Fourier transform (DFT) beamforming algorithms for space-time-frequency applications in wireless.

1. INTRODUCTION

This work presents modified variants, in a recursive format, of the Kahaner additive fast Fourier transform (FFT) algorithm. The variants are presented in Kronecker products algebra language. The language serves as a tool for the analysis, design, modification and implementation of the FFT variants on re-configurable field programmable gate array (FPGA) computational structures. The target for these computational structures are discrete Fourier transform (DFT) beamforming algorithms for space-time-frequency applications in wireless. When using Kroncker products algebra, a given FFT algorithm can be written as a decomposition of basic factors or mathematical expressions which we term functional primitives. This decomposition action establishes a one-one correspondence between a mathematical formulation of an algorithm and a given hardware computational structure such as an FPGA. Variants of a given mathematical formulation can be obtained using properties of Kronecker products algebra. These variants may satisfy certain design criteria such as pipelining, parallelism, data flow control, etc. In turn, each of these new variants will produce a different hardware implementation. The efficiency of each algorithm is evaluated when a cost function is imposed on the design criteria. We proceed to describe in detail a Kronecker decomposition for the Kahaner's FFT algorithm.

2. KRONECKER DECOMPOSITION OF KAHANER'S ALGORITHM

In his paper [18], D. K. Kahaner describes a procedure for factoring the Fourier matrix F_N when $N = p^\gamma$, p and γ any integers. Kahaner's factorization method produces, up to matrix factor expansion, what is commonly known as the Cooley-Tukey (C-T) decimation in frequency algorithm. In this section we describe Kahaner's algorithm in detail, and then present it a Kronecker product formulation. This will aid in the understanding of the Kronecker products language used to analyze other FFT algorithms later on.

2.1 Kahaner's Mathematical Formulation

Kahaner starts by defining the discrete Fourier transform of N equally spaced data points $x_k, k = 0,\ldots,N-1$:

$$F_r = \frac{1}{N}\sum_{k=0}^{N-1} x_k e^{-\frac{2\pi i}{N}rk} = \frac{1}{N}\sum_{k=0}^{N-1} x_k a^{rk},\ 0 \le r < N,\ a = e^{-\frac{2\pi i}{N}} \tag{2.1}$$

In matrix form,

$$\overline{F} = \frac{1}{N}A\overline{X}, \overline{F}^T = \begin{bmatrix} F_0 & F_1 & \cdots & F_{N-1} \end{bmatrix} \tag{2.2}$$

where A is the matrix:

$$A=\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & a & a^2 & \cdots & a^{(N-1)}\\ 1 & a^2 & a^4 & \cdots & a^{2(N-1)}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 1 & a^{N-1} & a^{2(N-1)} & \cdots & a^{(N-1)(N-1)}\end{bmatrix} \tag{2.3}$$

Kahaner proceeds to write a general expression for $F_r, r = 0,1,\ldots,N-1$:

$$F_r = \frac{1}{N}\sum_{k=0}^{N-1} x_k a^{rk} = \frac{1}{N}\sum_{k=0}^{M-1}\left\{\sum_{t=0}^{p-1} x_{k+tM} a^{r(k+tM)}\right\}, M = p^{\gamma-1}, r = 0,1,\ldots N-1 \tag{2.4}$$

Writing $r = pm + l$, the following expression is obtained,

$$F_r = F_{pm+l} = \frac{1}{N}\sum_{k=0}^{N-1} x_k a^{rk} = \frac{1}{N}\sum_{k=0}^{M-1}\left\{\sum_{t=0}^{p-1} x_{k+tM}\theta^{lt} a^{lk} a^{pmk}\right\}, \theta = e^{\frac{-2\pi i}{p}} \tag{2.5}$$

For each fixed l, the following vector is formed:

$$\left[F_1 \quad F_{p+1} \quad F_{2p+1} \quad \cdots \quad F_{(M-1)p+1}\right]^T, l = 0,1,\ldots,p-1 \tag{2.6}$$

After some algebraic manipulations, this vector is written as,

$$\frac{1}{N}\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & a^p & a^{2p} & \cdots & a^{(M-1)p}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 1 & a^{(M-1)p} & a^{2(M-1)p} & \cdots & a^{(M-1)(M-1)p}\end{bmatrix}\begin{bmatrix}\sum_{t=0}^{p-1} x_{tM}\theta^{lt}\\ \sum_{t=0}^{p-1} x_{1+tM}\theta^{1t}a^1\\ \vdots\\ \sum_{t=0}^{p-1} x_{(M-1)+tM}\theta^{1t}a^{1(M-1)}\end{bmatrix} \tag{2.7}$$

or

$$\begin{bmatrix}F_1\\ F_{p+1}\\ \vdots\\ F_{(M-1)p+1}\end{bmatrix} \equiv \frac{1}{-N} B\overline{X}_l^1, l = 0,\ldots,p-1 \tag{2.8}$$

By writing these p vectors $(l = 0,1,\ldots,p-1)$ in a column, the following result is obtained:

$$\overline{F}^{(1)} = \begin{bmatrix} \begin{bmatrix} F_0 \\ F_p \\ \vdots \\ F_{(M-1)p+0} \end{bmatrix} \\ \begin{bmatrix} F_1 \\ F_{p+1} \\ \vdots \\ F_{(M-1)p+1} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} F_{p-1} \\ F_{2p-1} \\ \vdots \\ F_{pM-p+p-1} \end{bmatrix} \end{bmatrix} \equiv \frac{1}{-N} \begin{bmatrix} B & & & \\ & B & & \\ & & \ddots & \\ & & & B \end{bmatrix} \begin{bmatrix} \overline{X}_0^{(1)} \\ \overline{X}_1^{(1)} \\ \vdots \\ \overline{X}_{p-1}^{(1)} \end{bmatrix} \tag{2.9}$$

where,

$$\begin{bmatrix} \overline{X}_0^{(1)} \\ \overline{X}_1^{(1)} \\ \vdots \\ \overline{X}_{p-1}^{(1)} \end{bmatrix} = \overline{X}^{(1)} = \Delta^{(0)} \overline{X} \tag{2.10}$$

and,

$$\Delta^{(0)} = \begin{bmatrix} I & I & \cdots & I \\ D & \theta D & \cdots & \theta^{P-1} D \\ D^2 & (\theta D)^2 & \cdots & (\theta^{P-1} D)^2 \\ \vdots & \vdots & \cdots & \vdots \\ D^{P-1} & (\theta D)^{P-1} & \cdots & (\theta^{P-1} D)^{P-1} \end{bmatrix}, D = diag\left[1, a, a^2, \ldots, a^{M-1}\right] \tag{2.11}$$

The vector $\overline{F}^{(1)}$ differs from the vector $\overline{F}$ by the permutation matrix π_0:

$$\overline{F} = \pi_0 \overline{F}^{(1)} = \frac{\pi_0}{-N} \begin{bmatrix} B & & & \\ & B & & \\ & & \ddots & \\ & & & B \end{bmatrix} \Delta^{(0)} \overline{X} \tag{2.12}$$

Since the matrix B has the general form of the matrix A, this result is generalized:

$$\overline{F} = \frac{\pi_0}{N}$$

$$\overline{F} = \frac{\pi_0}{N} hp_1 hp_2 \cdots hp_{\gamma-1} \begin{bmatrix} \Delta^{(\gamma-1)} & & \\ & \ddots & \\ & & \Delta^{(\gamma-1)} \end{bmatrix} \cdots \begin{bmatrix} \Delta^{(1)} & & \\ & \ddots & \\ & & \Delta^{(1)} \end{bmatrix} \left[\Delta^{(0)}\right] \overline{X} \tag{2.13}$$

$$= \frac{\pi_0}{N_1} 2 \cdots \gamma - 1 \begin{bmatrix} \Delta^{(\gamma-1)} & & \\ & \ddots & \\ & & \Delta^{(\gamma-1)} \end{bmatrix} \cdots \begin{bmatrix} \Delta^{(1)} & & \\ & \ddots & \\ & & \Delta^{(1)} \end{bmatrix} \left[\Delta^{(0)}\right]$$

where

$$hp_i = \begin{bmatrix} \pi_i & & & \\ & \pi_i & & \\ & & \ddots & \\ & & & \pi_i \end{bmatrix}, i = 1,2,\ldots,\gamma - 1 \tag{2.14}$$

$$i = \begin{bmatrix} \pi_i & & & \\ & \pi_i & & \\ & & \ddots & \\ & & & \pi_i \end{bmatrix}, i = 1,2,\ldots,\gamma - 1$$

$$\Delta^{(1)} = \begin{bmatrix} I & I & \cdots & I \\ D^{(j)} & \theta D^{(j)} & \cdots & \theta^{P-1} D^{(j)} \\ \vdots & \vdots & \cdots & \vdots \\ \left(D^{(j)}\right)^{P-1} & \left(\theta D^{(j)}\right)^{P-1} & \cdots & \left(\theta^{P-1} D^{(j)}\right)^{P-1} \end{bmatrix} \tag{2.15}$$

$$D^{(j)} = \begin{bmatrix} 1 & & & \\ & a^{p^j} & & \\ & & \ddots & \\ & & & a^{\left(p^{\gamma-j-1}-1\right)p^j} \end{bmatrix}, j = 0,1,\ldots,\gamma - 1, D^{\gamma-1} \equiv [1] \tag{2.16}$$

2.2 Kronecker Products Formulation

We proceed to describe this matrix factorization method in kronecker products form. We start by introducing the following definitions:

The matrix $P_{n,s}$, of order $n = r \cdot s$ is called the stride by s permutation matrix, and is defined by

$$P_{n,s} \cdot d = \left(d_0, d_s, d_{2s}, \ldots, d_1, d_{s+1}, \ldots, d_{(r-1)s+s-1}\right)^T \tag{2.17}$$

for

$$d = \left(d_0, d_1, \ldots, d_{s-1}, d_s, \ldots, d_{(r-1)s+s-1}\right)^T \tag{2.18}$$

The diagonal matrix $D_{n,s}$ of order s is defined by

$$D_{n,s} = diag\left[1, W_n, W_n^2, \ldots, W_n^{s-1}\right], W_n = e^{-2\pi i/n} \tag{2.19}$$

The twiddle factor (phase factor) matrix $D_{r,n/s}$ of order n/s :

$$T_{n,s}(r) = \sum_{0 \le j < s} \oplus D^j_{r,n/s} \tag{2.20}$$

If $n = r \cdot s$, then

$$T_{n,s}(n) = T_{n,s} = \sum_{0 \le j < s} \oplus D^j_{n,n/s} = \sum_{0 \le j < s} \oplus D^j_{n,r} \tag{2.21}$$

To arrive to a general form for the Fourier matrix F_N, $N = p^\gamma$, expressed in Kronecker products, we start with an expression for the Fourier matrix F_p and use this expression, and the Cooley-Tukey decimation in frequency algorithm expressed in Kronecker products, to obtain higher order Fourier matrices expressed in Kronecker products form:

$$F_N = F_{p^\gamma} = F_p = P_{p,1}\Delta^{(\gamma-1)} = T_{p,p}\Delta^{(\gamma-1)} = I_p\Delta^{(\gamma-1)}, \gamma = 1 \tag{2.22}$$

where

$$\Delta^{(\gamma-1)} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{(p-1)} \\ 1 & w^2 & w^4 & \cdots & w^{2(p-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & w^{p-1} & w^{2(p-1)} & \cdots & w^{(p-1)(p-1)} \end{bmatrix}, w = e^{-2\pi i/p} \tag{2.23}$$

The Cooley-Tukey decimation in frequency algorithm allows us to write F_{p^2} in the following form:

$$F_{p^2} = P_{p^2,p}\left(I_p \otimes F_p\right)T_{p^2,p}\left(F_p \otimes I_p\right) \tag{2.24}$$

Using

$$\Delta^{(\gamma-2)} = T_{p^2,p}\left(F_p \otimes I_p\right) \tag{2.25}$$

we get,

$$F_{p^2} = P_{p^2,p}\left(I_p \otimes F_p\right)\Delta^{(\gamma-2)} \tag{2.26}$$

where

$$T_{p^2,p}(r) = \sum_{0 \le j < s} \oplus D^j_{p,p}, D_{p,p} = \left[1, w_p, w_p^2, \ldots, w_p^{p-1}\right] \tag{2.27}$$

Using the expression for F_p given above, we obtain,

$$F_{p^2} = P_{p^2,p}\left(I_p \otimes P_{p,1}\right)\left(I_p \otimes \Delta^{(\gamma-1)}\right)\Delta^{(\gamma-2)} \tag{2.28}$$

For the matrix F_{p^3}, we write down again the expression for the Cooley-Tukey decimation in frequency algorithm:

$$F_{p^3} = P_{p^3,p^2}\left(I_p \otimes F_{p^2}\right)T_{p^3,p}\left(F_p \otimes I_{p^2}\right) \tag{2.29}$$

where,

$$T_{p^3,p}(r) = \sum_{0 \le j < s} \oplus D^j_{p^2,p}, D_{p^2,p} = \left[1, w_{p^2}, w_{p^2}^2, \ldots, w_{p^2}^{p-1}\right] \tag{2.30}$$

Using, and the expression given above for F_{p^2}, we get,

$$F_{p^3} = P_{p^3,p^2}\left(I_p \otimes P_{p^2,p}\right)\left(I_{p^2} \otimes P_{p,1}\right)\cdot\left(I_{p^2} \otimes \Delta^{(\gamma-1)}\right)\left(I_p \otimes \Delta^{(\gamma-2)}\right)\Delta^{(\gamma-3)} \tag{2.31}$$

Continuing in the same manner:

$$F_{p4} = P_{p^4,p3}\left(I_p \otimes F_{p^3}\right)T_{p^4,p}\left(F_p \otimes I_{p^3}\right) \tag{2.32}$$

Using

$$\Delta^{(\gamma-4)} = T_{p^4,p}\left(F_p \otimes I_{p^3}\right) \tag{2.33}$$

we get,

$$F_{p^4} = P_{p^4,p^3}\left(I_p \otimes P_{p^3,p^2}\right)\left(I_{p^2} \otimes P_{p^2,p}\right)\left(I_{p^3} \otimes P_{p,1}\right)\cdot\left(I_{p^3} \otimes \Delta^{(\gamma-1)}\right)\left(I_{p^2} \otimes \Delta^{(\gamma-2)}\right)\left(I_p \otimes \Delta^{(\gamma-3)}\right)\Delta^{(\gamma-4)} \tag{2.34}$$

In general, for a Fourier matrix $F_{p^{\gamma-k}}, 0 \le k < \gamma$, we write:

$$F_{p^{\gamma-k}} = P_{p^{\gamma-k}, p^{\gamma-k-1}}\left(I_p \otimes F_{p^{\gamma-k-1}}\right)T_{p^{\gamma-k},p}\left(F_p \otimes I_{p^{\gamma-k-1}}\right) \tag{2.35}$$

We, again, set

$$\Delta^{(\gamma-(\gamma-k))} = \Delta^{(k)} T_{p^{\gamma-k},p}\left(F_p \otimes I_{p^{\gamma-k-1}}\right) \tag{2.36}$$

where

$$T_{p^{\gamma-k},p} = \sum_{0 \le j < s} \oplus D^{j}_{p^{\gamma-k}, p^{\gamma-k-1}}, 0 \le k < \gamma; \tag{2.37}$$

and the identity

$$w_{p^{\gamma-k-1}} = e^{-2\pi i p\left(p^k / p^\gamma\right)} = \left(e^{-2\pi i p^k / p^\gamma}\right)^p = w^p_{p^{\gamma-k-1}}; \tag{2.38}$$

may be used to write down the elements of $D_{p^{\gamma-k}, p^{\gamma-k-1}}, 0 \le k < \gamma$

The general expression for $F_{p^{\gamma-k}}$ thus becomes:

$$F_{p^{\gamma-k}} = P_{p^{\gamma-k}, p^{\gamma-k-1}}\left(I_p \otimes P_{p^{\gamma-k-1}, p^{\gamma-k-2}}\right)\left(I_{p^2} \otimes P_{p^{\gamma-k-2}, p^{\gamma-k-3}}\right)\cdots$$
$$\cdots\left(I_{p^{\gamma-k-1}} \otimes P_{p,1}\right)\cdot\left(I_{p^{\gamma-k-1}} \otimes \Delta^{(\gamma-1)}\right)\left(I_{p^{\gamma-k-2}} \otimes \Delta^{(\gamma-2)}\right)\cdots\left(I_p \otimes \Delta^{(k+1)}\right)\Delta^{(k)} \tag{2.34}$$

3. CONCLUSIONS

This has presented modified variants, in a recursive format, of the Kahaner additive fast Fourier transform (FFT) algorithm. The variants are presented in Kronecker products algebra language. The language has been used as a tool for the analysis, design, modification and implementation of the FFT variants on re-configurable field programmable gate array (FPGA) computational structures.

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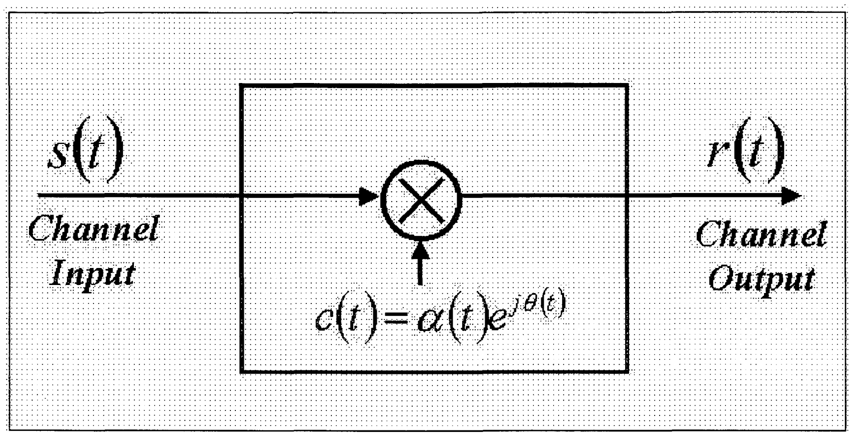

Figure 1. Multiplicative Channel Model

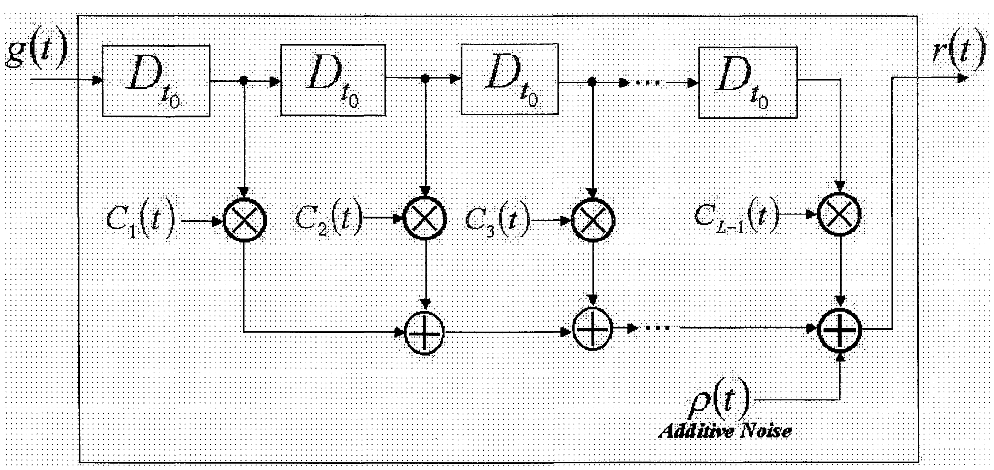

Figure 2. Tapped-Delay-Line Channel Model

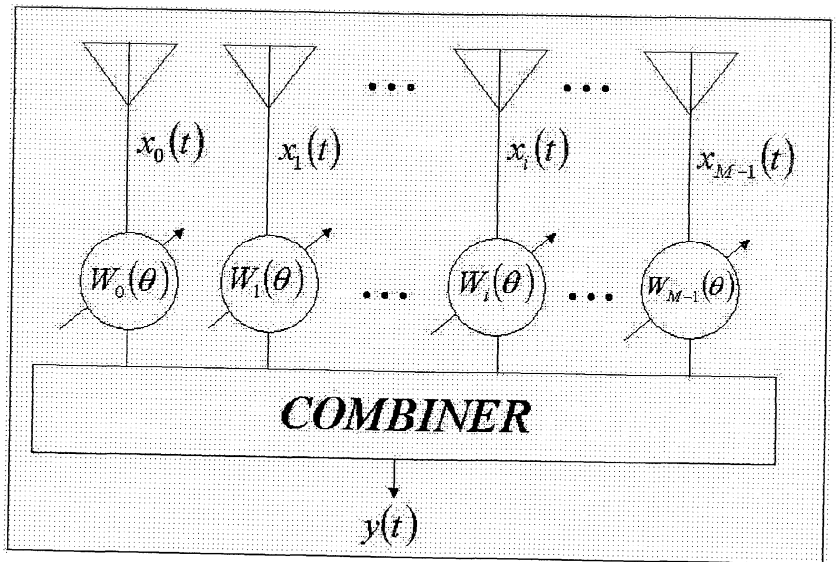

Figure 3. Basic Array Beamforming

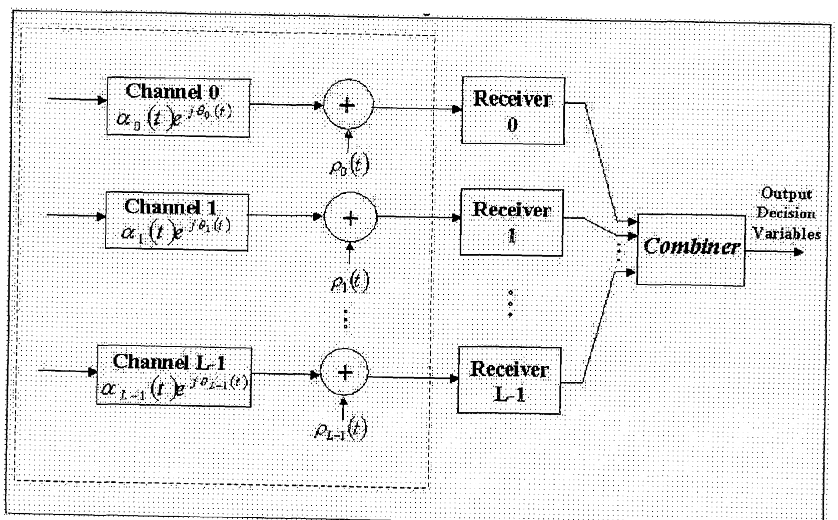

Figure 4. Binary Digital Communications Diversity Model