2. Use the following matrices to demonstrate properties of Kronecker products.

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} d_{00} & d_{01} \\ d_{10} & d_{11} \end{bmatrix}$$

Properties of the kronecker products:

a) $(AB) \otimes (CD) = (A \otimes C)(B \otimes D)$
b) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
c) $(A \otimes B)^T = A^T \otimes B^T$

a)

$$(AB) \otimes (CD)$$

$$\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \cdot \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} \otimes \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} \cdot \begin{bmatrix} d_{00} & d_{01} \\ d_{10} & d_{11} \end{bmatrix}$$

$$\begin{bmatrix} (a_{00}b_{00} + a_{01}b_{10}) & (a_{00}b_{01} + a_{01}b_{11}) \\ (a_{10}b_{00} + a_{11}b_{10}) & (a_{10}b_{01} + a_{11}b_{11}) \end{bmatrix} \otimes \begin{bmatrix} (c_{00}d_{00} + c_{01}d_{10}) & (c_{00}d_{00} + c_{01}d_{10}) \\ (c_{10}d_{00} + c_{11}d_{10}) & (c_{10}d_{01} + c_{11}d_{11}) \end{bmatrix}$$

(See next page to complete array of the kronecker product)

Making the first element of the array (0,0):

$$a_{00}b_{00}c_{00}d_{00} + a_{00}b_{00}c_{01}d_{10} + a_{01}b_{10}c_{00}d_{00} + a_{01}b_{10}c_{01}d_{10}$$

Rearranging terms:

$$a_{00}c_{00}b_{00}d_{00} + a_{00}c_{01}b_{00}d_{10} + a_{01}c_{00}b_{10}d_{00} + a_{01}c_{01}b_{10}d_{10}$$

Using the first column elements. We can see these match with the following array product:

$$\begin{bmatrix} a_{00}c_{00} & a_{00}c_{01} & a_{01}c_{00} & a_{01}c_{01} \\ \vdots & \vdots & \vdots\vdots & \\ & & & \\ a_{10}c_{10} & a_{10}c_{11} & a_{11}c_{10} & a_{11}c_{11} \end{bmatrix} \cdot \begin{bmatrix} b_{00}d_{00} & \dots & b_{01}d_{01} \\ b_{00}d_{10} & \dots & b_{01}d_{11} \\ b_{10}d_{00} & \dots & b_{11}d_{01} \\ b_{10}d_{10} & \dots & b_{11}d_{11} \end{bmatrix}$$

$$\begin{bmatrix} a_{00}c_{00} & a_{00}c_{01} & a_{01}c_{00} & a_{01}c_{01} \\ a_{00}c_{10} & a_{00}c_{11} & a_{01}c_{10} & a_{01}c_{11} \\ a_{10}c_{00} & a_{10}c_{01} & a_{11}c_{00} & a_{11}c_{01} \\ a_{10}c_{10} & a_{10}c_{11} & a_{11}c_{10} & a_{11}c_{11} \end{bmatrix} \cdot \begin{bmatrix} b_{00}d_{00} & b_{00}d_{01} & b_{01}d_{00} & b_{01}d_{01} \\ b_{00}d_{10} & b_{00}d_{11} & b_{01}d_{10} & b_{01}d_{11} \\ b_{10}d_{00} & b_{10}d_{01} & b_{11}d_{00} & b_{11}d_{01} \\ b_{10}d_{10} & b_{10}d_{11} & b_{11}d_{10} & b_{11}d_{11} \end{bmatrix}$$

This is the Kronecker product of:

$$(A \otimes C)(B \otimes D)$$

$$
\begin{bmatrix}
(a_{00}b_{00} + a_{01}b_{10})(c_{00}d_{00} + c_{01}d_{10}) & (a_{00}b_{00} + a_{01}b_{10})(c_{00}d_{01} + c_{01}d_{11}) & (a_{00}b_{01} + a_{01}b_{11})(c_{00}d_{00} + c_{01}d_{10}) & (a_{00}b_{01} + a_{01}b_{11})(c_{00}d_{01} + c_{01}d_{11}) \\
(a_{00}b_{00} + a_{01}b_{10})(c_{10}d_{00} + c_{11}d_{10}) & (a_{00}b_{00} + a_{01}b_{10})(c_{10}d_{01} + c_{11}d_{11}) & (a_{00}b_{01} + a_{01}b_{11})(c_{10}d_{00} + c_{11}d_{10}) & (a_{00}b_{01} + a_{01}b_{11})(c_{10}d_{01} + c_{11}d_{11}) \\
(a_{10}b_{00} + a_{11}b_{10})(c_{00}d_{00} + c_{01}d_{10}) & (a_{10}b_{00} + a_{11}b_{10})(c_{00}d_{00} + c_{01}d_{10}) & (a_{10}b_{01} + a_{11}b_{11})(c_{00}d_{00} + c_{01}d_{10}) & (a_{10}b_{01} + a_{11}b_{11})(c_{00}d_{01} + c_{01}d_{11}) \\
(a_{10}b_{00} + a_{11}b_{10})(c_{10}d_{00} + c_{11}d_{10}) & (a_{10}b_{00} + a_{11}b_{10})(c_{10}d_{01} + c_{11}d_{11}) & (a_{10}b_{01} + a_{11}b_{11})(c_{10}d_{00} + c_{11}d_{10}) & (a_{10}b_{01} + a_{11}b_{11})(c_{10}d_{01} + c_{11}d_{11})
\end{bmatrix}
$$

b)

$$A^{-1} \otimes B^{-1}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{11} & -a_{01} \\ -a_{10} & a_{00} \end{bmatrix}$$

$$B^{-1} = \frac{1}{\det B} \begin{bmatrix} b_{11} & -b_{01} \\ -b_{10} & b_{00} \end{bmatrix}$$

$$= \frac{1}{\det A} \begin{bmatrix} a_{11} & -a_{01} \\ -a_{10} & a_{00} \end{bmatrix} \otimes \frac{1}{\det B} \begin{bmatrix} b_{11} & -b_{01} \\ -b_{10} & b_{00} \end{bmatrix}$$

$$= \frac{1}{\det A} \frac{1}{\det B} \begin{bmatrix} +a_{11} \begin{bmatrix} b_{11} & -b_{01} \\ -b_{10} & b_{00} \end{bmatrix} & -a_{01} \begin{bmatrix} b_{11} & -b_{01} \\ -b_{10} & b_{00} \end{bmatrix} \\ -a_{10} \begin{bmatrix} b_{11} & -b_{01} \\ -b_{10} & b_{00} \end{bmatrix} & +a_{00} \begin{bmatrix} b_{11} & -b_{01} \\ -b_{10} & b_{00} \end{bmatrix} \end{bmatrix}$$

$$A^{-1} \otimes B^{-1} = \frac{1}{\det A} \frac{1}{\det B} \begin{bmatrix} +a_{11}b_{11} & -a_{11}b_{01} & -a_{01}b_{11} & +a_{01}b_{01} \\ -a_{11}b_{10} & +a_{11}b_{00} & +a_{01}b_{10} & -a_{01}b_{00} \\ -a_{10}b_{11} & +a_{10}b_{01} & +a_{00}b_{11} & -a_{00}b_{01} \\ +a_{10}b_{10} & -a_{10}b_{00} & -a_{00}b_{10} & +a_{00}b_{00} \end{bmatrix}$$

This result will be used later.

We are interested in the product $\det A \cdot \det B$.

$$\begin{aligned} \det A \cdot \det B &= (a_{00}a_{11} - a_{01}a_{10}) \cdot (b_{00}b_{11} - b_{01}b_{10}) \\ &= a_{00}b_{00}a_{11}b_{11} - a_{00}b_{01}a_{11}b_{10} - a_{01}b_{00}a_{10}b_{11} + a_{01}b_{01}a_{10}b_{10} \end{aligned}$$

Reordering the terms of the equation and multiply by -1 we obtain:

$$K = -a_{00}b_{11}a_{11}b_{00} + a_{01}b_{00}a_{10}b_{11} + a_{00}b_{01}a_{11}b_{10} - a_{01}b_{10}a_{10}b_{01}$$

We will try to take the other side of the test: $(A \otimes B)^{-1}$ the result of the Kronecker of A and B is a 4x4 matrix. Make a 4x4 inverse matrix is a really hard work, for this reason i will use the following Matlab program:

```
clear;
clc;
A = sym('A');
a00=sym('a00');
a01=sym('a01');
a10=sym('a10');
a11=sym('a11');
B = sym('B');
b00=sym('b00');
b01=sym('b01');
```

```
b10=sym('b10');
b11=sym('b11');
C = sym('C');
Cinv = sym('Cinv');
Cmat = sym('Cmat');
A=[ a00 a01 ; a10 a11];
B=[ b00 b01 ; b10 b11];
C=kron(A,B);
Cinv=inv(C)
K=-(-a00*b11*a11*b00+a01*b00*a10*b11+a00*b01*a11*b10-a01*b10*a10*b01)
Cmat=K.*Cinv;
simplify(Cmat)
```

The result of the program above is:

$$(A \otimes B)^{-1}(\det A \cdot \det B) = \begin{bmatrix} +a_{11}b_{11} & -a_{11}b_{01} & -a_{01}b_{11} & +a_{01}b_{01} \\ -a_{11}b_{10} & +a_{11}b_{00} & +a_{01}b_{10} & -a_{01}b_{00} \\ -a_{10}b_{11} & +a_{10}b_{01} & +a_{00}b_{11} & -a_{00}b_{01} \\ +a_{10}b_{10} & -a_{10}b_{00} & -a_{00}b_{10} & +a_{00}b_{00} \end{bmatrix}$$

Notice that the $\frac{1}{\det A \cdot \det B}$ is eliminated of the final result because the K in the Matlab Program. I do this in order to simplify the final Matlab expression.

The two ways arrive to the same result, thus the property is demonstrated.

c)

$$(A \otimes B)^T$$

$$\left(\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \otimes \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix} \right)^T$$

$$\begin{bmatrix} a_{00}b_{00} & a_{00}b_{01} & a_{01}b_{00} & a_{01}b_{01} \\ a_{00}b_{10} & a_{00}b_{11} & a_{01}b_{10} & a_{01}b_{11} \\ a_{10}b_{00} & a_{10}b_{01} & a_{11}b_{00} & a_{11}b_{01} \\ a_{10}b_{10} & a_{10}b_{11} & a_{11}b_{10} & a_{11}b_{11} \end{bmatrix}^T$$

$$\begin{bmatrix} a_{00}b_{00} & a_{00}b_{10} & a_{10}b_{00} & a_{10}b_{10} \\ a_{00}b_{01} & a_{00}b_{11} & a_{10}b_{01} & a_{10}b_{11} \\ a_{01}b_{00} & a_{01}b_{10} & a_{11}b_{00} & a_{11}b_{10} \\ a_{01}b_{01} & a_{01}b_{11} & a_{11}b_{01} & a_{11}b_{11} \end{bmatrix}$$

$$\begin{bmatrix} a_{00}\begin{bmatrix} b_{00} & b_{10} \\ b_{01} & b_{11} \end{bmatrix} & a_{10}\begin{bmatrix} b_{00} & b_{10} \\ b_{01} & b_{11} \end{bmatrix} \\ a_{01}\begin{bmatrix} b_{00} & b_{10} \\ b_{01} & b_{11} \end{bmatrix} & a_{11}\begin{bmatrix} b_{00} & b_{10} \\ b_{01} & b_{11} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} a_{00} & a_{10} \\ a_{01} & a_{11} \end{bmatrix} \otimes \begin{bmatrix} b_{00} & b_{10} \\ b_{01} & b_{11} \end{bmatrix}$$

This is the Kronecker product of:

$$A^T \otimes B^T$$

3. Demonstrate that $(F_{N_0} \otimes F_{N_1}) = (I_{N_0} \otimes F_{N_1}) \cdot (F_{N_0} \otimes I_{N_1})$ for specific values of N_0, N_1.

$N_0 = 3, N_1 = 2$

$$F_{N_0} = F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^1 & W_3^2 \\ 1 & W_3^2 & W_3^4 \end{bmatrix} \quad , \quad F_{N_1} = F_2 = \begin{bmatrix} 1 & 1 \\ 1 & W_2^1 \end{bmatrix}$$

Then:

$$F_3 \otimes F_2 = \begin{bmatrix} 1\begin{bmatrix} 1 & 1 \\ 1 & W_2^1 \end{bmatrix} & 1\begin{bmatrix} 1 & 1 \\ 1 & W_2^1 \end{bmatrix} & 1\begin{bmatrix} 1 & 1 \\ 1 & W_2^1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 1 \\ 1 & W_2^1 \end{bmatrix} & W_3^1\begin{bmatrix} 1 & 1 \\ 1 & W_2^1 \end{bmatrix} & W_3^2\begin{bmatrix} 1 & 1 \\ 1 & W_2^1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 1 \\ 1 & W_2^1 \end{bmatrix} & W_3^2\begin{bmatrix} 1 & 1 \\ 1 & W_2^1 \end{bmatrix} & W_3^4\begin{bmatrix} 1 & 1 \\ 1 & W_2^1 \end{bmatrix} \end{bmatrix}$$

$$F_3 \otimes F_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_2^1 & 1 & W_2^1 & 1 & W_2^1 \\ 1 & 1 & W_3^1 & W_3^1 & W_3^2 & W_3^2 \\ 1 & W_2^1 & W_3^1 & W_3^1W_2^1 & W_3^2 & W_3^2W_2^1 \\ 1 & 1 & W_3^2 & W_3^2 & W_3^4 & W_3^4 \\ 1 & W_2^1 & W_3^2 & W_3^2W_2^1 & W_3^4 & W_3^4W_2^1 \end{bmatrix}$$

Now taken the other side of the equation : $(I_{N_0} \otimes F_{N_1}) \cdot (F_{N_0} \otimes I_{N_1})$

$$I_{N_0} \otimes F_{N_1} = I_3 \otimes F_2$$

$$I_3 \otimes F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & W_2^1 \end{bmatrix}$$

$$I_3 \otimes F_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & W_2^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & W_2^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & W_2^1 \end{bmatrix}$$

$$F_{N_0} \otimes I_{N_1} = F_3 \otimes I_2$$

$$F_3 \otimes I_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^1 & W_3^2 \\ 1 & W_3^2 & W_3^4 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_3 \otimes I_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & W_3^1 & 0 & W_3^2 & 0 \\ 0 & 1 & 0 & W_3^1 & 0 & W_3^2 \\ 1 & 0 & W_3^2 & 0 & W_3^4 & 0 \\ 0 & 1 & 0 & W_3^2 & 0 & W_3^4 \end{bmatrix}$$

Now multiplying:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & W_2^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & W_2^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & W_2^1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & W_3^1 & 0 & W_3^2 & 0 \\ 0 & 1 & 0 & W_3^1 & 0 & W_3^2 \\ 1 & 0 & W_3^2 & 0 & W_3^4 & 0 \\ 0 & 1 & 0 & W_3^2 & 0 & W_3^4 \end{bmatrix}$$

$$(I_3 \otimes F_2) \cdot (F_3 \otimes I_2) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_2^1 & 1 & W_2^1 & 1 & W_2^1 \\ 1 & 1 & W_3^1 & W_3^1 & W_3^2 & W_3^2 \\ 1 & W_2^1 & W_3^1 & W_3^1 W_2^1 & W_3^2 & W_3^2 W_2^1 \\ 1 & 1 & W_3^2 & W_3^2 & W_3^4 & W_3^4 \\ 1 & W_2^1 & W_3^2 & W_3^2 W_2^1 & W_3^4 & W_3^4 W_2^1 \end{bmatrix}$$

This result is the same that the $(F_3 \otimes F_2)$