Non-Uniform Discrete Short-Time Fourier Transform A Goertzel Filter Bank Approach

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Abstract

It is well known that the discrete Short Time Fourier Transform (STFT) can be considered from the perspective of a Discrete Fourier Transform (DFT) taken over short time sections of the signal (the window is fixed) or from the perspective of a filtering operation at a given frequency (the frequency is fixed). The latter approach is typically useful when only a few frequencies of interested are required or when the window used to segment the data has infinite length. For such windows the implementation requires a bank of recursive filters (IIR). When finite length windows are used, the bank of filters is non-recursive (FIR). In such case the use of a suitable approximation by IIR filters could eventually reduce the computational cost. We are proposing a mixed approach where the spectrum of each finite short time section of the signal is found by realizing a DFT through a bank of IIR Goertzel Filters centered at specified frequencies. This approach allows computing the time varying spectrum at precisely the frequencies of interest. In fact, the Goertzel algorithm is used to implement the Non-Uniform Discrete Fourier Transform (NDFT). Within the NDFT framework the estimation of the spectrum at the frequency of interests it is not conditioned to the requirement that the DFT index, k, be an integer. We have termed this implementation of the discrete STFT the "Non-Uniform Discrete Short Time Fourier Transform" (NSTFT). A MATLAB program was written using this technique and validated. Future work includes computational cost analysis, synthesis issues and a viability study regarding the use of the so-called Sliding Goertzel DFT for the implementation of a discrete STFT.

1. Introduction

1.1 A Goertzel filter bank approach.

We are proposing an approach to the implementation of a discrete STFT where the spectrum of each finite short time section of the signal is estimated by realizing the DFT through a bank of IIR Goertzel Filters. The technique has all the advantages inherent to the IIR Goertzel filters, including the capacity of doing sample by sample processing, and can estimate the spectrum at the exact frequencies of interest. This is because the Goertzel algorithm is suitable for the implementation of a non-uniform discrete Fourier transform (NDFT). The NDFT it is not conditioned to the requirement that the DFT index, k, be an integer. We have termed this realization "Non-Uniform Discrete Short Time Fourier Transform" (NSTFT).

1.2 Organization of the paper

First we review the DFT approach followed by the filtering approach to the realization of the discrete STFT. The Goertzel algorithm is then derived and proposed, in conjunction with the Non-Uniform DFT as a third alternative to the implementation of the STFT.

2. Traditional Approaches to the Implementation of the Discrete STFT.

2.1 The discrete-time STFT and the discrete STFT

We define the discrete time STFT at time n as

$$X(n,w) = \sum_{m=-\infty}^{\infty} x[m]w[n-m]e^{-jwm} \qquad (1)$$

where m is the discrete time variable. Usually n is defined as n = LN where N is the window length. The amount of overlap between adjacent windows depends on L.

If the frequency is discretized and the result taken over one period [Nawab87], the discrete STFT can be defined as

$$X(n,k) = \sum_{m=-\infty}^{\infty} x[m]w[n-m]e^{-j2pkm/N} R_N[k] \quad (2)$$

where $R_N[k] = u(k) - u(k-N)$

2.2 DFT approach.

This approach consists in taking the DFT of the short time sections of the signal. In other words at each time $n=n_o$ we consider a particular short time section of the signal. We are therefore fixing the window and estimating the spectrum of that particular short time section over N, equally spaced, points over the unit circle. The definition is as follows,

$$X(n_o,k) = \sum_{m=-\infty}^{\infty} x[m]w[n_o-m]e^{-j2pkm/N} R_N[k] \quad (3)$$

where $x_w[m] = x[m]w[n_o-m]$ is a particular short-time section of the signal. A block diagram realization of this perspective is depicted in Figure 1.

$$X(n_o,k) = \sum_{m=-\infty}^{\infty} x_w[m]e^{-j2pkm/N} R_N[k] \quad (4)$$

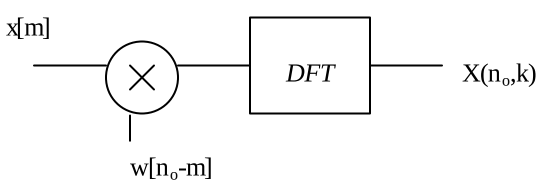

Fig. 1. Discrete STFT Implementation from a DFT Point of View.

2.3 Filtering approach.

This perspective is completely different. Here the frequency, instead of the window, is fixed at a certain value ? = $?_o$. The discrete time STFT expression in (1) becomes

$$X(n,w_o) = \sum_{m=-\infty}^{\infty} x[m]w[n-m]e^{-jw_o m} \quad (5)$$

This new relation depicts the signal modulated by a complex exponential and convolved with the window.

$$X(n,w_0) = \left[x[n]e^{-jw_0 n}\right] * w[n] \quad (6)$$

Its block diagram realization is

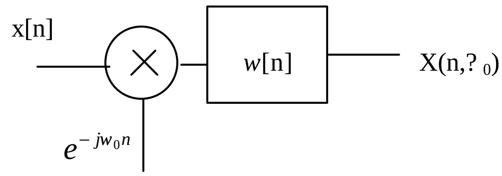

Fig. 2. Discrete-Time STFT Implementation from a Filtering Point of View.

The window becomes the impulse response and determines whether the filter is IIR or FIR.

A slightly different realization can be achieved if equation (6) is rewritten as

$$X(n,w_0) = e^{-jw_0 n}[x[n] * w[n]e^{+jw_0 n}] \quad (7)$$

Where its block diagram realization becomes

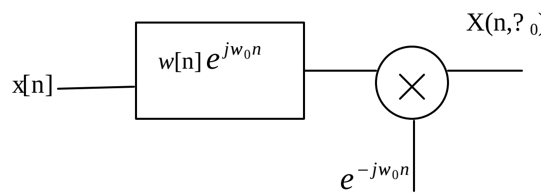

Fig. 3. Discrete-Time STFT Implementation from a Filtering Point of View. A Modified Version.

Their discrete-frequency counterpart can be obtained by substituting $?_o$ by $2pk_o/N$. Figure 4 illustrates the realization in Figure 3 adapted to the discrete-frequency case.

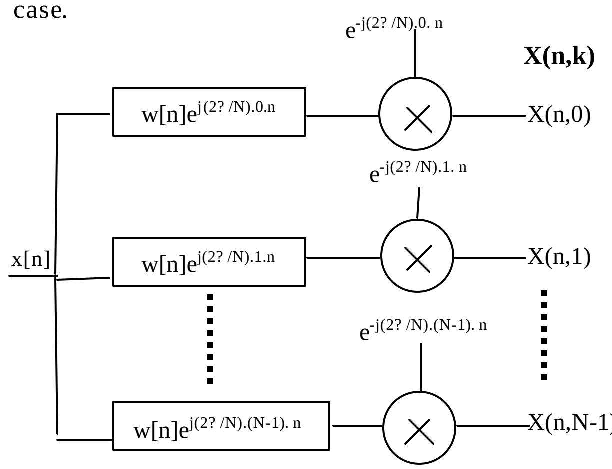

Fig. 4. Discrete STFT Implementation from a Filtering Point of View.

For each k we have a different filter, therefore the complete realization of a Discrete STFT requires the use of a filter bank. A detailed discussion of both approaches can be found in [Quatieri01].

3. The Non-Uniform DFT.

The nonuniform discrete Fourier transform (NDFT) of a discrete time signal of length N is defined as [Bagchi99]

$$X[z_k] = \sum_{n=0}^{N-1} x[n] z_k^{-n} \quad k = 0,1, \ldots, N-1 \qquad (8)$$

where $z_0, z_1, \ldots, z_{N-1}$ are different points in the z-plane arbitrarily located.

For the study of the spectrum of a signal the N-1 points in the z-plane can be taken distinctly but arbitrarily over the unit circle. If they are taken equally spaced over the unit circle starting at ? = 0 the NDFT reduces to DFT (9),

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2pk}{N}n} \quad k = 0,1, \ldots, N-1 \qquad (9)$$

It is well known that in the DFT, for a given N, the frequency of interest not always maps to a k integer (10). In those cases our spectral estimates will be in error.

$$k = \frac{f_{interest}}{f_S}.N \qquad (10)$$

The following expression allows us to compute (9) at arbitrarily spaced frequencies over the unit circle.

$$\frac{k}{N} = \frac{f_{interest}}{f_S} \qquad (11)$$

Substituting (11) in (9) gives a Non-Uniform DFT evaluated over N frequencies of interest over the unit circle.

Note that the filter bank in Figure 4 lends itself to the implementation of a non-uniform discrete STFT. In such case we have to make the substitution indicated in (11) for all the frequencies of interest. As a matter of fact MATLAB uses such realization to compute the spectrogram at arbitrary frequencies. Being able to estimate the spectral content of a signal at exactly the frequency of interest it is especially important in applications such as DTMF detection and many others.

4. A Goertzel Filter Bank Approach to the Implementation of the DFT

In order to derive the Goertzel algorithm we start with the definition of the DFT,

$$X[k] = W_N^{-kN} \sum_{m=0}^{N-1} x[m] W_N^{km} \qquad (12)$$

Multiplying (12) by $W_N^{-kN} = 1$ gives

$$X[k] = W_N^{-kN} \sum_{m=0}^{N-1} x[m] W_N^{km} \qquad (14)$$

which can be written as

$$X[k] = \sum_{m=0}^{N-1} x[m] W_N^{-k(N-m)} \qquad (15)$$

If we let N=n in (15) we obtain an expression $y[n]$ that has the form of a convolution,

$$y_k[n] = \sum_{m=0}^{N-1} x[m] W_N^{-k(n-m)} \qquad (16)$$

$$\text{where } h_k[n] = W_N^{-kn} \qquad (17)$$

Looking at (15) and (16) we conclude that

$$X[k] = y_k[n]\big|_{n=N} \qquad (18)$$

In other words the Nth output of the filter in (17) gives the k_{th} bin of the DFT. In order to find a realization for this filter we take the Z transform of (17)

$$H_k(z) = \sum_{n=0}^{\infty} W_N^{-kn} z^{-n} = \sum_{n=0}^{\infty} \left[W_N^{-k} z^{-1}\right]^n \qquad (18)$$

$$H_k[z] = \frac{1}{1 - W_N^{-k} z^{-1}}, \quad |z| > \left|W_N^{-k}\right| \qquad (19)$$

Multiplying numerator and denominator by the conjugate of the denominator we obtain an expression with real coefficients in the denominator,

$$H_k(z) = \frac{1 - W_N^{k} z^{-1}}{1 - 2\cos\left(\frac{2p}{N}k\right) z^{-1} + z^{-2}} \qquad (20)$$

The system function in (20) can be implemented using a direct form II realization (Fig. 5.),

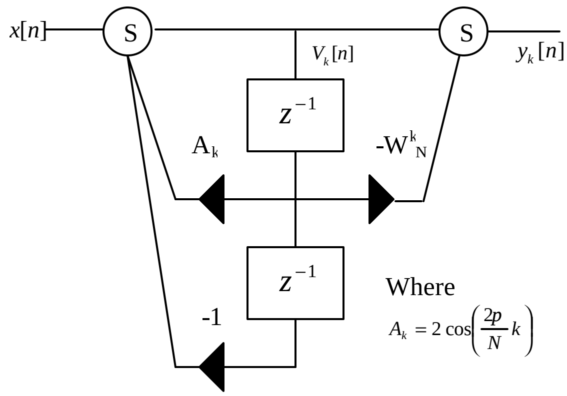

Figure 5. Goertzel Filter, Direct Form II Realization.

Note that (18) the feedforward loop needs to be computed only once (for the Nth sample). This realization lends itself to a sample by sample implementation suitable to real time applications. When only the square magnitude of the filter output is needed (such as in the spectrogram) the calculations can be further simplified [Bagchi99]. If for each k we define a different filter, then we have a filter bank suitable for the computation of the DFT.

In order to compute a NDFT at *arbitrarily* spaced frequencies of interest over the unit circle it suffices to use (11) in the filter coefficients shown in Figure 5.

5. A Goertzel Filter Bank Approach to the Implementation of the Discrete STFT

We are proposing a combined approach in which the window is fixed and within each short time section of the signal the frequencies of interest are also fixed. The NDFT of the short time section is implemented using a Goertzel filter bank centered at such frequencies.

When we select N *equally* spaced frequencies over the unit circle our MATLAB implementation (Fig. 6.) of the NSTFT algorithm (Fig. 7.) reduces, as expected, to the traditional spectrogram.

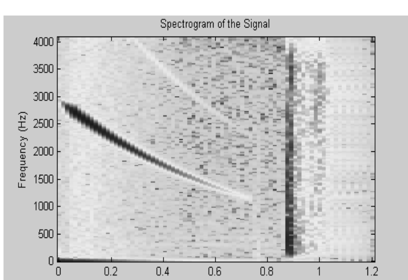

Figure 6. Spectrogram of the Signal Splat.

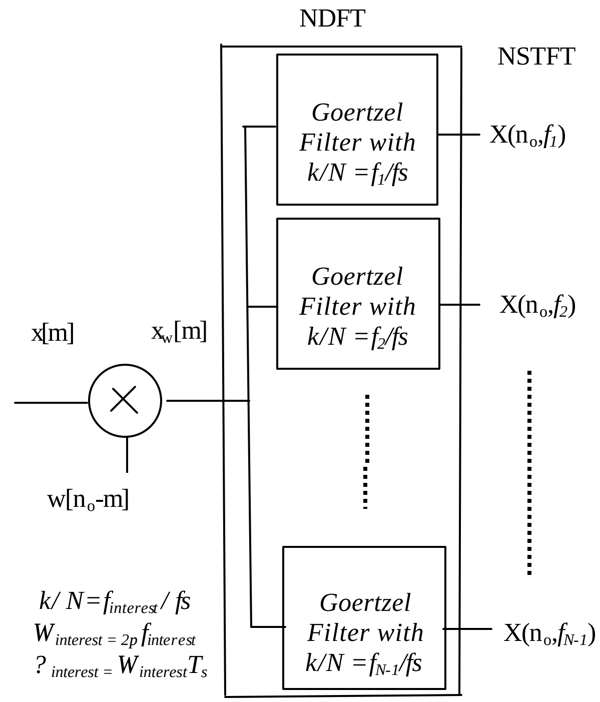

Figure 7. Realization of a Non-Uniform Discrete STFT using a recursive Goertzel Filter Bank

6. Conclusions and Future Work

We have presented a third alternative to the two traditional approaches regarding the implementation of a discrete STFT. The proposed approach uses a bank of IIR Goertzel filters and is amenable to the implementation of a non-uniform STFT. Future work includes computational cost analysis, comparison with the traditional filtering approach and synthesis issues. We will also continue working on DTMF tone detection using the NDFT and the Goertzel Algorithm.

References

[Nawab87] Nawab S.H. and Quatieri T. F., “Short Time Fourier Transform”, Chapter in Advanced Topics in Signal Processing, J.S. Lim and A.V. Oppenheim, eds, Prentice Hall, Englewood Cliffs, NJ, October 1987.

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[Bagchi99] Bagchi S. and Mitra S. K., “ The NonUniform Discrete Fourier Transform and its Applications in Signal Processing”, Kluwer Academic Publishers, Norwell Massachusetts 1999.