Quantum Algorithm for N-Queens problem

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Abstract

We study the application of quantum computing to the Queens problem. The presented quantum algorithm for this combinatorial search problem allows significant reduction of number of iteration.

1 Introduction

In the game of chess the queen piece can move in any direction (horizontal, vertical and diagonal) and jump over any number of spaces. The n-queens problem is a classical combinatorial problem related with the distribution of n number of queens on an nxn chess board. The queens are distributed in a such way that no two of them lie in the same row, column, or diagonal, so none of them can attack each other. The problem is famous for simple demonstrations of advanced computation methods like neural networks, genetic algorithms and parallel algorithms . In this paper we will demonstrate that a solution of the n-queens problem can be found efficiently by a quantum algorithm.

It is well known that the theoretical quantum computer is based on quantum physical principles. This is in contrast with the basic priciples of current computer technology, which are based on classic non-quantum principles.

We use a quantum bit as a piece of information, which is described by a superposition of basic states:

$$|\Psi\rangle = C_0|0\rangle + C_1|1\rangle,$$

where $|\Psi\rangle$ is a wave function from the Schroedinger equation, $C_0|0\rangle$ and $C_1|1\rangle$ are "0" and "1" states. The probability of finding our q-bit, $|\Psi\rangle$, in state "0" is given by $|C_0|^2$. Likewise, $|C_1|^2$gives the probability of state "1" at any given time. This ability of a q-bit to exist in a blend of all of its allowed states is not the only quantum quality that makes such a system ideal for the field of computation. We will find that when we ask a collection of q-bits to produce a result, and there is more than one possible result, the collection experiences quantum interference. This means that different results will interfere with each other, destructively or constructively, having the effect of damping or amplifying each particular result of all the allowed states. For the solution of our problem we will use the Grover iteration. The Grover method is a database search scheme for which a maximum of $\sqrt{N}$ (where $N = n^n$) iterations are needed to obtain the desired result.

2 Theoretical Formulation

For the solution of our problem we define a quantum state of all the possible solution states. We do this by using the Hadamard operator over a classical register with n entries, in order to produce a quantum register. Then we define two unitary operators, which we combine according to the Grover Iteration scheme. One of our unitary operators is called the **oracle**. The **oracle** is a subroutine that quickly evaluates a function $f_\omega(x)$ that checks a proposed solution to problem. After we submit a query of x to the oracle it will tell us whether $x = \omega$ or not. It returns the value of the function $f_\omega(x)$, with

$$\begin{aligned} f_\omega(x) &= 0, x \neq \omega \\ f_\omega(x) &= 1, x = \omega. \end{aligned}$$

We may write the oracle in the following way:

$$U_\omega = 1 - 2|\omega\rangle\langle\omega|$$

This oracle flips the sign of $|\omega\rangle$, but acts trivially on any state orthogonal to $|\omega\rangle$.

As a preparation for the Grover algorithm, we first set up a state $|s\rangle$. This state is created by using the Hadamard transformation for each q-bit initially in state $|x = 0\rangle$. Then we get an equally weighted superposition of all computational basis states.

$$|s\rangle = \frac{1}{\sqrt{N}} \sum |\omega\rangle$$

By combining the unknown reflectionU_ω performed by the oracle with the following reflection

$$Us = 2|s\rangle\langle s| - 1$$

which also preserves $|s\rangle$, but flips the sign of any vector orthogonal to $|s\rangle$ we obtain the Grover iteration operator, in the following form :

$$R_{grov} = U_s U_\omega$$

After performing this quantum step once we find one particular solution, then by using symmetry properties we get all its supplementary solutions. The complete algorithm comprises a repeated application of this one step, until all solutions have been found.

3 Research Results

To find the solution of the problem we have defined a quantum state with all possible solutions. Taking into account that a register with all the possible states represents too many undesirable states, we prepare a special quantum state $|\Psi\rangle$ in the following way. We first prepare a sequence of n registers,

$$\begin{aligned} R1 &= (1000\ldots\ 0) \\ R2 &= (0100\ldots\ 0) \\ &. \\ &. \\ &. \\ Rn &= (0000\ldots\ 1), \end{aligned}$$

where n is the number of queens. Solutions of the n-queens problem represent permutations of this sequence. We store these classical registers in a quantum memory register. That will consists in of equally weighted superposition of n states representing each classical input. Then we make n 'q-nits' as we can call them by analogy with the two-state q-bits,

$$\begin{aligned} |U_1\rangle &= \frac{1}{\sqrt{n}}(|R_1\rangle + |R_2\rangle + \ldots + |R_n\rangle) \\ |U_2\rangle &= \frac{1}{\sqrt{n}}(|R_1\rangle + |R_2\rangle + \ldots + |R_n\rangle) \\ &\cdots \\ |U_n\rangle &= \frac{1}{\sqrt{n}}(|R_1\rangle + |R_2\rangle + \ldots + |R_n\rangle) \end{aligned}$$

which we combine into a quantum register

$$|\Psi\rangle = \frac{1}{\sqrt{N}}(|U_1\rangle \otimes |U_2\rangle \otimes \ldots \otimes |U_n\rangle).$$

or

$$|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_x (|x\rangle$$

This q-register represents all possible states for n queens. In the same way we prepare the $|s\rangle$ state for the unitary operator

$$U_s = 2|s\rangle\langle s| - 1.$$

Then for the oracle we define the function $f_w(x)$ with the following conditions:

1.Row check

$$\sum_{j,i=0}^{n-1} R_{i+nj} = 1$$

2. Column check

$$\sum_{i,j=0}^{n-1} R_{i+nj} = 1$$

3. Diagonal check

$$
\begin{aligned}
1 &= \sum_{j=0}^{(n-1)-i} R_{i+(n+1)j}, \\
i &= 0,1,\ldots,(n-1) \\
1 &= \sum_{i=0}^{i \bmod (n-1)} R_{i+(n+1)j}, \\
j &= n2-1, n2-2, \ldots, n2-(n-1) \\
1 &= \sum_{j=0}^{i} R_{i+(n-1)j}, \\
i &= 1,\ldots,(n-1) \\
1 &= \sum_{i=0}^{(n+1)-i \bmod (n+1)} R_{i+(n-1)j}, \\
j &= n^2-1, n^2-2, \ldots, n^2-(n-1)
\end{aligned}
$$

Now we can define the unitary operator for the oracle

$$U_\omega : |x\rangle \to (-1)^{f_\omega(x)}|x\rangle$$

or

$$U_\omega = 1 - 2|\omega\rangle\langle\omega|.$$

After we have both unitary operators we define the Grover unitary transform $R_{grov} = U_s U_w$. By applying several times the Grover iteration algorithm we can obtain all possible solutions.

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