Control of AC Motors

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DRAFT

Contents

Preface

This book is intended to be an exposition of the modeling and control of AC motors, specifically, the induction, PM synchronous, stepper, and switched reluctance motors. The particular emphasis here is on techniques used for high performance applications, that is, applications that require both rapid and precise control of position, speed and/or torque. Traditionally, DC motors were reserved for high performance applications (positioning systems, rolling mills, traction drives, etc.) because of their relative ease of control compared to AC motors. However, with the advances in computing and power electronics, AC motors continue to replace DC motors in high performance applications. The exposition here is to carefully derive the mathematical models of the AC machines and show how these mathematical models are used to design control algorithms that achieve high performance.

Electric machines are a particularly fascinating application of basic electricity and magnetism. The presentation here relies heavily on these basic concepts from Physics to develop the models of the motors. Specifically, Faraday's Law $\xi = -d\phi/dt$ where $\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}$, the magnetic force law $\vec{\mathbf{F}} = i\vec{\ell} \times \vec{\mathbf{B}}$ (or, $\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$), Gauss's Law $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$, Ampere's Law $\oint \vec{\mathbf{H}} \cdot d\vec{\ell} = i_{\text{free}}$, the relationship between $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$, properties of magnetic materials, etc. are reviewed in detail and used extensively to derive the currently accepted nonlinear differential equation models of the various AC motors. The authors have attempted to make the modeling assumptions as clear as possible and to show that the magnetic and electric fields satisfy Maxwell's equations (as, of course, they must). Many elementary books on electric machines tend to make a superficial use of electricity and magnetism in regards to electric machines. Instead, they rely on stating the equivalent circuit models of the machine and analyzing them ad nauseam. However, the equivalent circuit is a result of making a linear approximation to a nonlinear differential equation model of the motor. Consequently, the emphasis here is on the derivation of the nonlinear model based on basic electricity and magnetism. The derivation of the corresponding equivalent circuit assuming steady-state conditions is then straightforward.

Electric machines also provide fascinating examples to illustrate concepts from electromagnetic field theory (in contrast to electricity and magnetism). In particular, how the electric and magnetic fields change as one goes between reference frames that are in relative motion can be nice illustrated using AC machines. Optional sections show how the electric and magnetic fields change as one goes between a coordinate system attached to the stator to a coordinate system that rotates with the rotating magnetic field produced by the stator currents or a frame attached to the rotor. Also given in an optional section is the derivation of the azimuthal electric and magnetic fields.

This is also a book on the control of AC machines based on their differential equation models. With the notable exception of the sinusoidal steady-state analysis of the induction motor in Chapter 7, very little attention is given to the classical equivalent circuits as these models are valid only in steady-state. Instead, the differential equation models are used as the basis to develop the notions of field-oriented control, input-output linearization, flux observers, least-squares identification methods, state feedback trajectory tracking, etc. This is a natural result of the emphasis here on high performance control methods (e.g., field-oriented control) as opposed to classical methods (e.g., V/f , slip control etc.)[1].

There are of course many good books in this areas of electric machines and their control. The authors owe a debt of gratitude to Professor W. Leonhard for his book [1] from which they were educated in the modeling and control of electric drives. The present book is much narrower in focus with an emphasis on the modeling and operation of the motors based on elementary classical physics and an emphasis on high performance control methods. The book by Krause [2] is clear and complete in its derivation of the mathematical models of electric machines while C.B.Gray [3] makes quite an effort to present electromagnetic theory in the context of electric machines. A comprehensive treatment using Simulink® to simulate electric machinery in given in C-M Ong's book [4]. The books by S.J. Chapman [5], Woodson & Melcher [6], Matsch [7], Krause and Wasynczuk [8], Mohan [9] and Slemon & Straughn [10] are good elementary books on machines.

The beautifully written textbooks *Physics* by the Physical Science Curriculum Study [11], *Physics* by D. Halliday and R. Resnick [12], *Principles of Electrodynamics* by M. Schwartz [13] and *Electromagnetic Fields* by R.K. Wangness [14] are used as references for the theory of electricity and magnetism.

This book borrows from these above works and hopefully adds its own contribution to the literature on AC machines.

Part one of the book consists of the first three chapters and presents a detailed review of the basic electricity and magnetism as well as an introduction to control that will be used extensively in the remaining chapters. Specifically:

Chapter 1 reviews the basic ideas of electricity and magnetism that are needed in the later chapters to model AC motors. In particular, the notions of magnetic fields, magnetic materials, magnetic force and Faraday's law

[1] The classical methods are discussed, but the high performance methods are covered in detail.

are reviewed by using them to derive the ideal model of a DC motor.

Chapter 2 provides an elementary introduction to the control ideas required for the high performance control of electric machines. An elementary presentation is given of state feedback control, speed observers and identification theory as applied to DC motors to setup the reader for the later chapters.

These first two chapters are elementary in nature and were written to be accessible to undergraduates. The reason for this is the fact that typically control engineers do not have a lot of background in modeling of machines while power/electric-machine engineers do not usually have a background in basic state-space concepts of control theory. Consequently, it is hoped that these two chapters can bring the reader up to speed in these areas.

Chapter 3 goes into the modeling of magnetic materials in terms of their use in electric machines. The fundamental result of this chapter is the modification of Ampere's Law $\oint_C \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 i$ so that it is valid in the presence of magnetic material. This introduces the notion the field $\vec{\mathbf{H}}$ and its relationship to $\vec{\mathbf{B}}$ to obtain the more general version of Ampere's Law $\oint \vec{\mathbf{H}} \cdot d\vec{\ell} = i_{\text{free}}$. All of this requires a significant discussion of the modeling of magnetic materials. The approximation that $\vec{\mathbf{H}}$ is zero in magnetic materials is discussed and shown how this approximation and Ampere's Law can be used to find the $\vec{\mathbf{B}}$ in the air gap of AC machines. Also presented is Gauss's Law for $\vec{\mathbf{B}}$ which leads to the notion of conservation of flux, as well as the fact that the normal component of $\vec{\mathbf{B}}$ must be continuous across the boundary between air and magnetic material. This chapter should be read, but the reader should not get "bogged down" in the chapter. Rather, the main results should be remembered.

Part two consists of the chapters 4 through 12 and presents the modeling and control of AC motors. Specifically:

Chapter 4 uses the results of Chapters 1&3 to explain how a radially directed rotating magnetic field can be established in the airgap of AC machines. In particular, the notion of sinusoidally wound turns (phase windings) is explained and then Ampere's Law is used to show that a sinusoidal (spatially) distributed radial magnetic field is established in the airgap by the currents in the phase windings.

Chapter 5 explains the fundamental physics behind the working of induction and synchronous machines. Specifically, this chapter uses a simplified model of the induction motor and shows how voltages and currents are induced in the rotor loops by the rotating magnetic field established by the stator currents. Then it is shown how torque is produced on these induced currents by the same stator rotating magnetic field that induced them. Similarly, the synchronous motor is analyzed to show how the stator rotating magnetic field produces torque on the rotor.

Chapter 6 derives the differential equation mathematical models that characterize two phase induction and synchronous machines. These models are the accepted models seen throughout the literature.

Chapter 7 This chapter presents the derivation of the models of three phase AC machines and their two-phase equivalent models. The classical steady-state analysis of induction motors is also presented.

Chapter 8 covers the control of induction motors presenting both field-oriented control and input-output linearization control. The notion of flux observers, field weakening, speed estimation based on position measurements as well as parameter identification methods are all discussed.

Chapter 9 covers the control of synchronous motors describing field-oriented control, field weakening, speed estimation as well as identification methods.

Chapter 10 covers the modeling and control of Switched Reluctance motors.

Chapter 11 covers advanced topics including sensorless control of induction motors

Chapter 12 covers the modeling and control of PM stepper motors.

Note to the reader: Sections marked with an asterisk * can be omitted without loss of continuity.

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Part I

Preliminaries and Background

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Chapter 1 reviews the basic ideas of electricity and magnetism that are needed in the later chapters to model AC motors. In particular, the notions of magnetic fields, magnetic materials, magnetic force and Faraday's law are reviewed by using them to derive the ideal model of a DC motor. A fundamental reference for this chapter is the beautifully written book by Halliday and Resnick [12]. An advanced treatment of Electromagnetic Theory is given in another beautifully written book by R.K. Wangness [14].

Chapter 2 provides an elementary introduction to the control ideas required for the high performance control of electric machines. An elementary presentation is given of state feedback control, speed observers, identification theory applied to DC motors to setup the reader for the later chapters.

These first two chapters are elementary in nature and were written to be accessible to undergraduates. The reason for this is the fact that typically control engineers do not have a lot of background in modeling of machines while power/electric-machine engineers do not usually have a background in basic state-space concepts of control theory. Consequently, it is hoped that these two chapters can bring the reader up to speed in these areas.

Chapter 3 goes into the modeling of magnetic materials in terms of their use in electric machines. The fundamental result of this chapter is Ampere's Law $\oint \vec{\mathbf{H}} \cdot d\vec{\ell} = i_{\text{free}}$ which requires a significant background in the modeling of magnetic materials. The approximation that $\vec{\mathbf{H}}$ is zero in magnetic materials is discussed and shown how this approximation and Ampere's Law can be used to find the $\vec{\mathbf{B}}$ in the air gap of AC machines. Also presented is Gauss's Law for $\vec{\mathbf{B}}$ which leads to the notion of conservation of flux, as well as the fact that the normal component of $\vec{\mathbf{B}}$ must be continuous across the boundary between air and magnetic material. This chapter should be read, but the reader should not get "bogged down" in the chapter. Rather, the main results should be remembered.

1

The Physics of the DC Motor

The principles of operation of a Direct Current (DC) motor are presented based on fundamental concepts from electricity and magnetism contained in any basic physics course. The DC motor is used as a concrete example for reviewing the concepts of magnetic fields, magnetic force, Faraday's Law and induced emf's that will be used throughout the remainder chapters for the modeling AC motors. Consequently, this chapter is far from being an exposition in the design and modeling of DC motors. All of the Physics concepts referred to in this chapter are contained in the book *Physics* by Halliday and Resnick [12].

1.1 Magnetic Force

Motors work on the basic principle that magnetic fields produce forces on wires carrying a current. In fact, this experimental phenomenon is what is used to define the magnetic field. If one places a current carrying wire between the poles of a magnet as in the figure below, a force is exerted on the wire. *Experimentally*, the magnitude of this force is found to be proportional to both the amount of current in the wire and to the length of the wire that is between the poles of the magnet. That is, $F_{\text{magnetic}} \propto \ell i$. The direction of the magnetic field $\vec{\mathbf{B}}$ at any point is defined to be the direction that a small compass needle would point at that location. This direction is indicated by arrows in Figure 1.1 below.

Further experiments show that if the wire is parallel to the $\vec{\mathbf{B}}$ field (rather than perpendicular as in the figure), then no force is exerted on the wire. If the wire is at some angle θ with respect to $\vec{\mathbf{B}}$, then the force is proportional to the component of the wire that is perpendicular to $\vec{\mathbf{B}}$. That is, with θ the angle between ℓ and $\vec{\mathbf{B}}$, so that $\ell_\perp = \ell \sin(\theta)$ is the component of $\vec{\ell}$ perpendicular to $\vec{\mathbf{B}}$, it is found experimentally that $F_{\text{magnetic}} \propto \ell_\perp i$. In words, the magnetic force is proportional to the amount of current in the wire and to the length of wire perpendicular to $\vec{\mathbf{B}}$. Based on these experimental results, the strength or magnitude of the *magnetic induction field* $\vec{\mathbf{B}}$ is then defined to be the constant of proportionality, that is,

$$B = |\,\vec{\mathbf{B}}\,| \;\triangleq\; \frac{F_{\text{magnetic}}}{\ell i}$$

where F_{magnetic} is the magnetic force, i is the current, and ℓ is the length of wire perpendicular to the magnetic field carrying the current. That is,

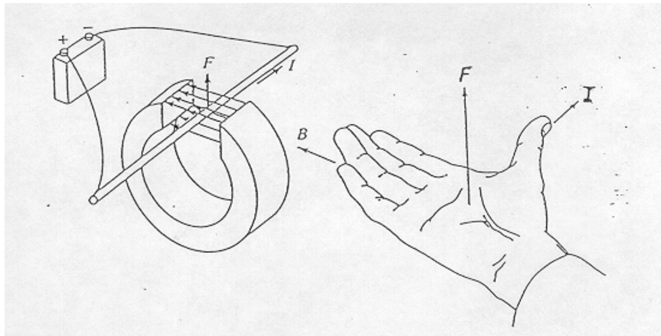


FIGURE 1.1. Magnetic Force Law (PSSC)

B is the proportionality constant so that $F_{\text{magnetic}} = i\ell B$.

When the $\vec{\mathbf{B}}$ field is perpendicular to the current carrying wire, the direction of the force can be determined using the right-hand rule: Using your right hand, point your fingers in the direction of the magnetic field and your thumb in the direction of the current. Then the direction of the force is out of your palm.

A more general way to relate the magnetic field $\vec{\mathbf{B}}$ to the force it produces is as follows: Let $\vec{\ell}$ denote a vector whose magnitude is the length ℓ of the wire in the magnetic field and whose direction is defined as the positive direction of current in the bar, then the magnetic force on the bar of length ℓ carrying the current i is given by

$$\begin{aligned} \vec{\mathbf{F}}_{\text{magnetic}} &= i\vec{\ell} \times \vec{\mathbf{B}} \\ \left|\vec{\mathbf{F}}_{\text{magnetic}}\right| &= i\ell B \sin(\theta) \\ &= i\ell_{\perp} B \qquad (\ell_{\perp} = \ell \sin(\theta)) \\ &= i\ell B_{\perp} \qquad (B_{\perp} = B \sin(\theta) \end{aligned}$$

where $B_{\perp} = B\sin(\theta)$ is just the component of $\vec{\mathbf{B}}$ perpendicular to the wire[1].

Example *A Linear DC Machine*

Consider the simple linear DC machine as in [5] where a sliding bar rests on a simple circuit consisting of two rails. An external magnetic field is going through the loop of the circuit up out of the page indicated by the $\otimes$ in the plane of the loop. Closing the switch results in a current flowing around the circuit and the external magnetic field produces a force on the bar which is free to move. The force on the bar is now computed.

The magnetic field is constant and points into the page (indicated by $\otimes$) Written in vector notation, $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$ where $B > 0$. By the right hand rule,

[1] Motors are designed so that the current carrying wire is perpendicular to the external magnetic field.

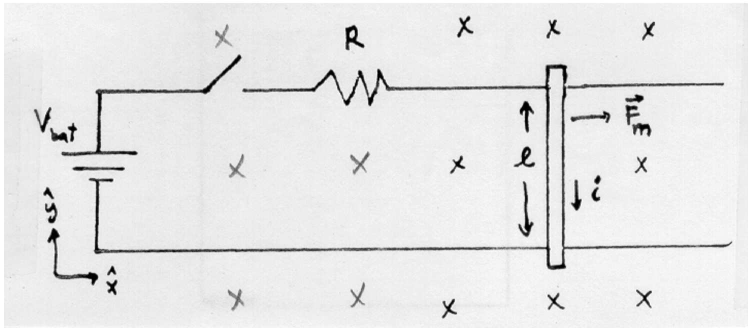


FIGURE 1.2. Linear DC motor

the magnetic force on the sliding bar points to the right. Explicitly, with $\vec{\ell} = -\ell\hat{\mathbf{y}}$ the force is given by

$$\begin{aligned}\vec{\mathbf{F}}_{\text{magnetic}} &= i\vec{\ell} \times \vec{\mathbf{B}} = i(-\ell\hat{\mathbf{y}}) \times (-\mathbf{B}\hat{\mathbf{z}}) \\ &= i\ell B\hat{\mathbf{x}}.\end{aligned}$$

To find the equations of motion for the bar, let f be the coefficient of viscous-friction of the bar so that the friction force is given by $F_f = -f dx/dt$. Then, with m_ℓ denoting the mass of the bar, Newton's law gives,

$$i\ell B - f dx/dt = m_\ell d^2x/dt^2$$

In order to solve for $x(t)$ the current $i(t)$ must be known. Assuming the switch has been closed for $t < 0$, $i(0) = V_{bat}/R$. However, it turns out that the current does *not* stay at this constant value, but decreases due to electromagnetic induction. This will be explained later.

1.2 Single-Loop Motor

As a first step to modeling a DC motor, a simplistic single-loop motor is considered. It is first shown how torque is produced and then how the current in the single-loop can be reversed (commutated) every half turn to keep the torque constant.

1.2.1 Torque Production

Consider the magnetic system in Figure 1.3 below [5], where a cylindrical core is cut out of a block of a permanent magnetic and replaced with a soft iron core. The term "soft" iron refers to the fact that material is easily magnetized (a permanent magnet is referred to as "hard" iron).

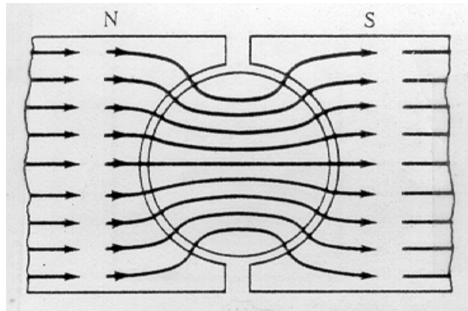


FIGURE 1.3. Soft iron cylindrical core placed inside a hollowed out Permanent Magnet to produce a radial magnetic field in the airgap

An important property of magnetic materials is that the magnetic field at the surface of such materials tends to be normal (perpendicular) to the surface. Consequently, the cylindrical shape of the soft iron core and stator field magnet has the effect of making the field in the air gap *radially* directed and furthermore it is reasonably constant (uniform) in magnitude. A mathematical description of the magnetic field in the air gap due to the permanent magnet is simply

$$\begin{aligned}\vec{\mathbf{B}} &= +B\hat{\mathbf{r}} \text{ for } 0<\theta<\pi \\ &= -B\hat{\mathbf{r}} \text{ for } \pi<\theta<2\pi\end{aligned}$$

where $B>0$ is the magnitude or strength of the magnetic field and θ is an arbitrary location in the air gap[2].

Figure 1.4 shows a rotor loop added to the magnetic system of Figure 1.3. The torque on this rotor loop is now calculated by considering the magnetic forces on sides a and a' of the loop [7] (On the other two sides of the loop, i.e., the front & back sides, the magnetic field has negligible strength so that no significant force is produced on these sides). As illustrated in Figure 1.4, the rotor angular position is taken to be the angle θ_R from the vertical to side a of the rotor loop. Figure 1.5 shows the cylindrical coordinate system used in Figure 1.4. Here $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}$ denote unit cylindrical coordinate vectors. The unit vector $\hat{\mathbf{z}}$ points along the rotor axis (into the paper in Figure 1.4), $\hat{\boldsymbol{\theta}}$ is in the direction of increasing θ and $\hat{\mathbf{r}}$ is in the direction of increasing r. Referring back to Figure 1.4, for $i>0$, the current in side a of the loop is going into the page (denoted by $\otimes$) and then comes out of the page (denoted by $\odot$) on side a'. Thus, on side a, $\vec{\ell}=\ell_1\hat{\mathbf{z}}$ (as $\vec{\ell}$ points in the direction of positive current flow) and the magnetic force $\vec{\mathbf{F}}_{\text{side } a}$ on side a

[2] Actually it will be shown in a later chapter that the magnetic field must be of the form $\vec{\mathbf{B}}=B\frac{r_0}{r}\hat{\mathbf{r}}$ in the airgap, that is, it varies as $1/r$ in the airgap. However, as the airgap is small, the $\vec{\mathbf{B}}$ field is essentially constant across the airgap.

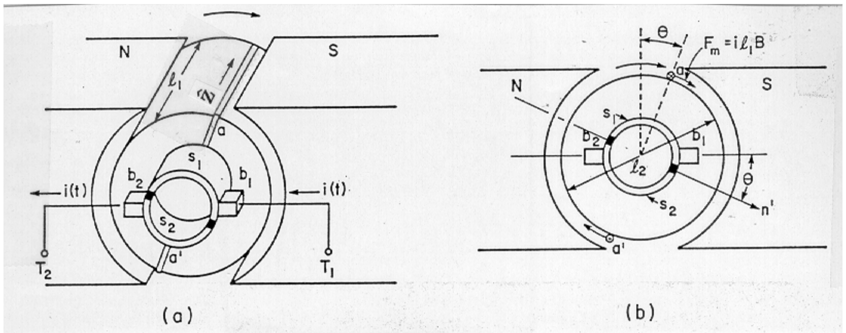


FIGURE 1.4. Single-Loop Motor

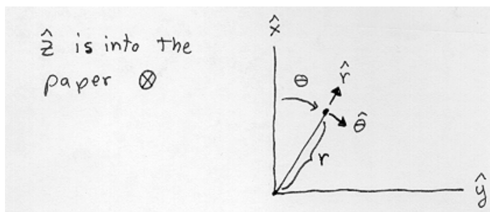


FIGURE 1.5. Cylindrical coordinate system used in Figure 1.4.

is then

$$\begin{aligned}\vec{\mathbf{F}}_{\text{side } a} &= i\vec{\ell} \times \vec{\mathbf{B}} \\ &= i(\ell_1 \hat{\mathbf{z}}) \times (B\hat{\mathbf{r}}) \\ &= i\ell_1 B\hat{\boldsymbol{\theta}}\end{aligned}$$

which is tangential to the motion as shown. The resulting torque is

$$\begin{aligned}\vec{\boldsymbol{\tau}}_{\text{side } a} &= (\ell_2/2)\hat{\mathbf{r}} \times \vec{\mathbf{F}}_{\text{side } a} \\ &= (\ell_2/2) i\ell_1 B\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \\ &= (\ell_2/2) i\ell_1 B\hat{\mathbf{z}}\end{aligned}$$

which is constant. Similarly, the magnetic force on side a' of the rotor loop is

$$\begin{aligned}\vec{\mathbf{F}}_{\text{side } a'} &= i\vec{\ell} \times \vec{\mathbf{B}} \\ &= i(-\ell_1 \hat{\mathbf{z}}) \times (-B\hat{\mathbf{r}}) \\ &= i\ell_1 B\hat{\boldsymbol{\theta}}\end{aligned}$$

so that the corresponding torque is then

$$\begin{aligned}\vec{\boldsymbol{\tau}}_{\text{side } a'} &= (\ell_2/2)\hat{\mathbf{r}}\times\vec{\mathbf{F}}_{\text{side } a'}\\ &= (\ell_2/2)i\ell_1 B\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}\\ &= (\ell_2/2)i\ell_1 B\hat{\mathbf{z}}\end{aligned}$$

The total torque on the rotor loop is then

$$\begin{aligned}\vec{\boldsymbol{\tau}}_m &= \vec{\boldsymbol{\tau}}_{\text{side } a} + \vec{\boldsymbol{\tau}}_{\text{side } a'}\\ &= 2(\ell_2/2)i\ell_1 B\hat{\mathbf{z}}\\ &= \ell_1\ell_2 B i\hat{\mathbf{z}}\end{aligned}$$

As the vector $\hat{\mathbf{z}}$ is along the axis of rotation so is the torque. In scalar form,

$$\tau_m = K_T i \text{ where } K_T = \ell_1\ell_2 B$$

The force is proportional to the strength of magnetic field $\vec{\mathbf{B}}$ in the air gap due to the permanent magnet. In order to increase the strength of the magnetic field in the air gap, the permanent magnet can be replaced with a soft iron material and then wire wound around the periphery of the magnetic material as shown in Figure 1.6. This winding is referred to as the *field winding* and the current it carries is called the *field current*. Typically, the strength of the magnetic field in the air gap is then proportional to the field current i_f at lower current levels ($B = K_f i_f$) and then saturates as the current increases. This may be written as $B = f(i_f)$ where $f(\cdot)$ is a saturation curve satisfying $f(0) = 0, f'(0) = K_f$ as shown in the figure below.

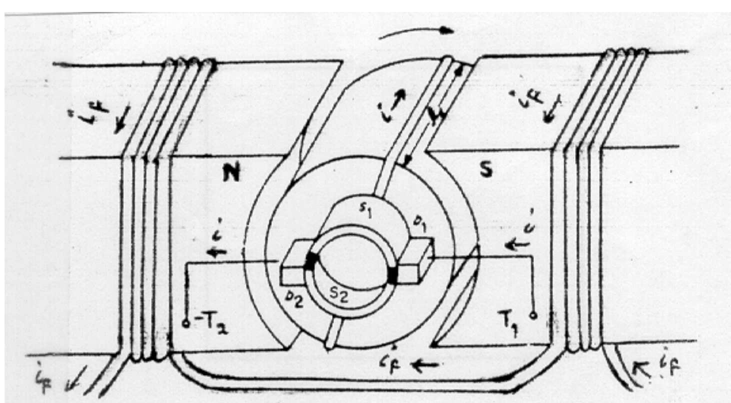


FIGURE 1.6. DC Motor with a Field Winding

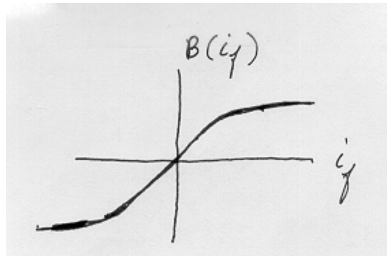


FIGURE 1.7. Radial magnetic field strength in the air gap as a function of field current.

1.2.2 Commutation of the Single-Loop Motor

The above derivation for the torque as $\tau_m = K_T i$ assumes that the current in the side of the rotor loop under the south pole face is into the page and the current in the side of the loop under the north pole face is out of the page. In order to make this assumption valid, the direction of the current in the loop must be changed as the rotor angle θ_R goes from $-\pi$ to 0 to π (see Figures 1.8a-d), . The process of changing the direction of the current is referred to as *commutation* and is done at $\theta_R = 0$ and $\theta_R = \pi$ through the use of the slip rings *s1*, *s2* and brushes *b1*, *b2*. The slip rings are rigidly attached to the loop and thus rotate with it. The brushes are fixed in space with the slip rings making a sliding contact with the brushes as the loop rotates. To see how the commutation of the current is accomplished using the brushes and slip rings, consider the sequence of figures below.

As shown in Figure 1.8a, the current goes through brush *b1* into the slip ring *s1*. From there, it travels down (into the page $\otimes$) side a of the loop, comes back up side a' (out of the page $\odot$) into the slip ring *s2* and finally, out the brush *b2*. Note that side a of the loop is under the south pole face while side a' is under the north pole face.

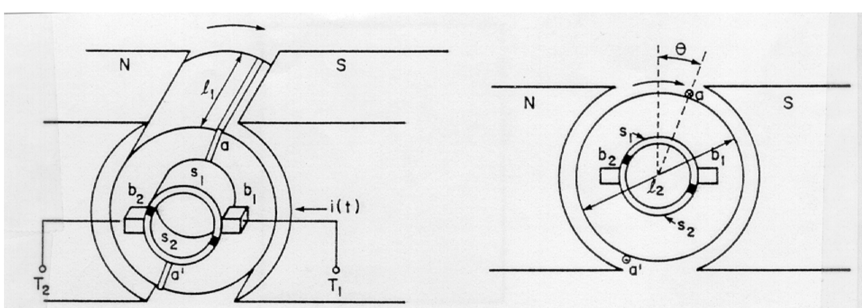


FIGURE 1.8. $0 < \theta_R < \pi$

Figure 1.8b shows the rotor loop just before commutation where the same comments as in Figure 1.8 apply.

Figure 1.8c shows that when $\theta_R = \pi$, the slip rings at the ends of the loop are shorted together by the brushes forcing the current in the loop to drop to zero.

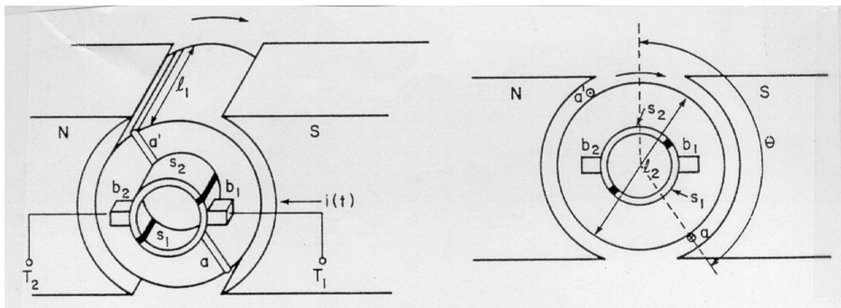


Figure 1.8b Rotor loop just prior to commutation where $0 < \theta_R < \pi$.

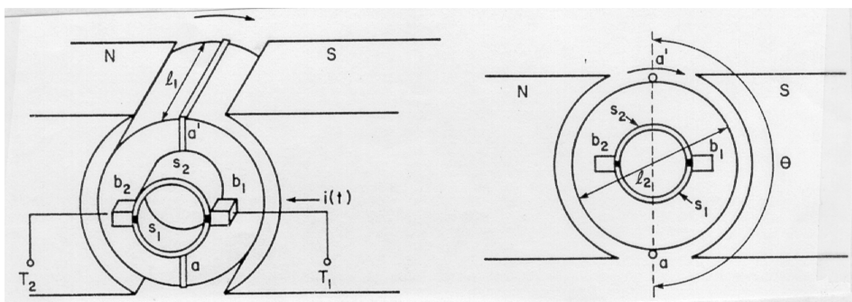


Figure 1.8c The ends of the rotor loop are shorted when $\theta_R = \pi$

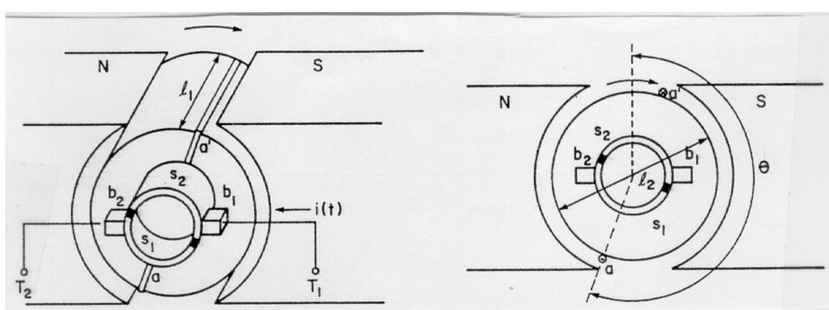


Figure 1.8d Rotor loop just after commutation where $\pi < \theta_R < 2\pi$

As shown in Figure 1.8d with $\pi < \theta_R < 2\pi$, the current is now going through brush *b1* into slip ring *s2*. From there, the current travels down (into the page $\otimes$) side a' of the loop and comes back up (out of the page $\odot$) side a. In other words, the current has *reversed* its direction in the loop from that in Figures 1.8a,1.8b. This is precisely what is desired, since side a is now under the north pole face and side a' is under the south pole face. As a result of the brushes and slip rings, the current direction in the loop is reversed every half-turn.

Remark *Voltage and Current Limits*

The amount of voltage v_a that may be applied to the input terminals T_1, T_2 of the motor is limited by capabilities of the amplifier supplying the voltage, that is, $|v_a| \leq V_{\max}$. Let $u(t)$ be the voltage commanded to the amplifier, then using the notation $sat(\cdot)$ to denote the saturation function (see Figure 1.9), $v_a = sat\left(K_a u(t)\right)$ indicates the dependence of v_a on $u(t)$.

In addition, there is a limit to the amount of current the rotating loop can handle before overheating or causing problems with commutation as previously mentioned. There are usually two current limits given. The first one is called the *continuous* current rating/limit and is the amount of current the motor can handle if left in use indefinitely. That is, the amount of heat dissipated in the rotor windings due to Ohmic losses is being taken away by thermal conduction through the brushes and thermal convection with the air so as to be in a thermal equilibrium. The second rating is the *peak* current rating/limit and is the amount of current the motor can handle for short (typically a few seconds or less) periods of time. Of course, the amount of current the amplifier can actually put out determines whether or not the peak current can be attained. The peak current is usually much higher than the continuous current rating.

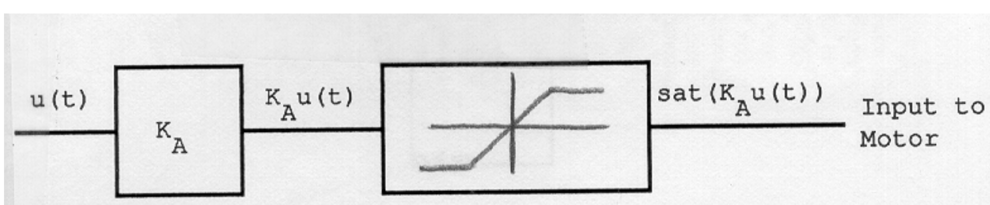


FIGURE 1.9. Saturation model of amplifier

1.3 Faraday's Law and Induced Electromotive Force (emf)

A changing flux within a loop produces an induced *electromotive force (emf)* ξ in the loop according to Faraday's law[3]. That is,

$$\xi = -d\phi/dt$$

where

$$\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}$$

is the flux in the loop and S is any surface with the loop as its boundary. The surface element $d\vec{\mathbf{S}}$ is a vector whose magnitude is the differential (small) element of area dS and whose direction is normal (perpendicular) to the surface element. As there are two possibilities for the normal to the surface, one must be chosen in a consistent manner as explained below. Depending on the particular normal chosen, a convention is then used to characterize positive and negative directions of travel around the boundary.

Sign Convention For Travel Around Surface Boundaries

In figure 1.10 below, the normal direction is taken to be up in the positive z direction so that $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, $dS = dxdy$ resulting in $d\vec{\mathbf{S}} = dxdy\hat{\mathbf{z}}$

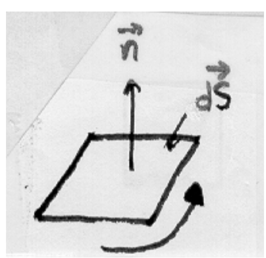


FIGURE 1.10. Positive direction of travel around a surface element with the normal up

In figure 1.11, the normal direction is taken to be down in the negative z direction so that $\hat{\mathbf{n}} = -\hat{\mathbf{z}}, dS = dxdy$ resulting in $d\vec{\mathbf{S}} = -dxdy\hat{\mathbf{z}}$

[3] ξ is the Greek letter Xi and is pronounced "Ki".

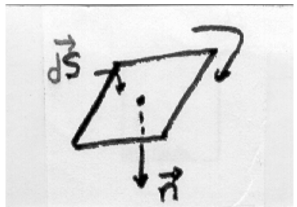


FIGURE 1.11. Positive direction of travel around a surface element with the normal down

As illustrated, the vector differential element of surface area $d\vec{\mathbf{S}}$ is defined to be a vector whose magnitude is the area of the differential area element and whose direction is normal to the surface. One may choose either normal (as long as it is a continuous vector field on the surface) and the corresponding direction of positive travel is then determined.

Connecting Two Surfaces Together

Two surface elements may be connected together and travel around the total surface defined as shown below.

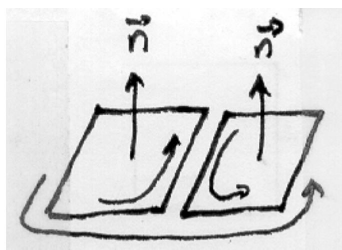


FIGURE 1.12. Positive direction of travel around two joined surface elements

The interpretation of positive and negative values of the induced electromotive force ξ is now explained. Faraday's law says,

$$\xi = -d\phi/dt$$

where

$$\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}$$

If ξ is positive, the induced emf will force current in the positive direction of travel around the surface while if ξ is negative, the induced emf will force current in the opposite direction. As illustrated in the next two exercises, this sign convention for Faraday's Law is just a precise mathematical way of describing Lenz's Law: *In all cases of electromagnetic induction, an induced voltage will cause a current to flow in a closed circuit in such a direction* *that the magnetic field which is caused by that current will oppose the change that produced the current* (see pages 873-877 of [12] for this version of Lenz's law).

1.3.1 Back EMF in a Linear DC Machine

The back emf in the linear DC machine [5] is now computed.

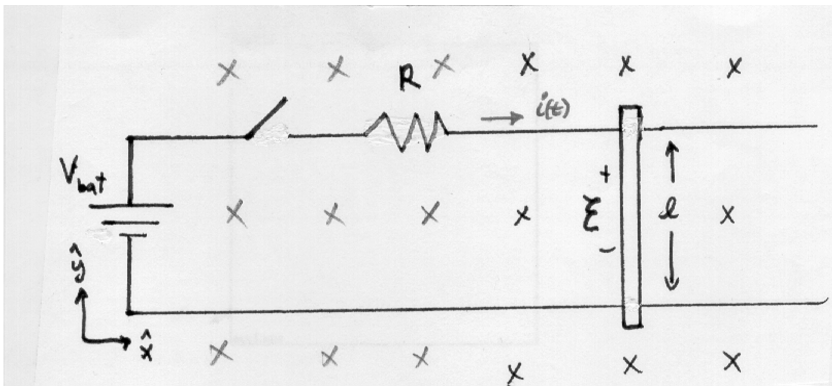


FIGURE 1.13.

The magnetic field is constant and points into the page, i.e., $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$, where $B > 0$. The magnetic force on the bar is $\vec{\mathbf{F}}_{\text{magnetic}} = i\ell B\hat{\mathbf{x}}$. To compute the induced voltage, let $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ be the normal to the surface so that $d\vec{\mathbf{S}} = dxdy\hat{\mathbf{z}}$ where $dS = dxdy$. Then

$$\begin{aligned}\phi &= \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \\ &= \int_0^\ell \int_0^x (-B\hat{\mathbf{z}}) \cdot (dxdy\hat{\mathbf{z}}) \\ &= \int_0^\ell \int_0^x -Bdxdy \\ &= -B\ell x\end{aligned}$$

The induced ("back") emf is therefore given by

$$\xi = -d\phi/dt = -d(-B\ell x)/dt = B\ell v$$

In the flux computation, the normal for the surface was taken to be in $+\hat{\mathbf{z}}$ direction which means that the positive direction of travel around the surface is counter-clockwise around the loop. That is, the sign conventions for battery voltage V_{bat} and the back-emf ξ are opposite which shows that the back-emf $\xi = B\ell v$ is *opposing* the applied battery voltage V_{bat}.

Remark $\phi = -B\ell x$ is the flux in the circuit due to the *external* magnetic field $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$. There is also a flux $\phi_L = Li$ due to the current in the circuit. For this example, the self-inductance is small, and so one sets $L = 0$. (See the next example below.)

Electromechanical Energy Conversion

As the (back) emf Blv opposes the current i, electrical power is being absorbed by this back emf. Specifically, the electrical power absorbed by the back emf is $i\xi = iB\ell v$ while the mechanical power produced is $F_{\text{magnetic}}v = i\ell Bv$. That is, the electrical power absorbed by the back emf reappears as mechanical power, as it must for energy conservation. Another way to view this is to note that $V_{bat}i$ is the electrical power delivered by the battery, and as $V_{bat} - B\ell v = Ri$, one may write

$$\begin{aligned} V_{bat}i &= Ri^2 + i(B\ell v) \\ &= Ri^2 + F_{\text{magnetic}}v \end{aligned}$$

That is, the power from the source $V_{bat}i$ is dissipated as heat in the resistance R and the rest converted into mechanical power.

Equations of Motion for the Linear DC Machine

The equations of motion for the bar in the linear DC machine are now derived. With the self-inductance L taken to be zero, m_ℓ the mass of the bar, f the coefficient of viscous-friction,

$$\begin{aligned} V_{\text{bat}} - B\ell v &= Ri \\ m_\ell \frac{dv}{dt} &= i\ell B - fv \end{aligned}$$

Eliminating the current i,

$$m_\ell \frac{d^2x}{dt^2} = \ell B(V_{\text{bat}} - B\ell v)/R - fv = -\left(\frac{B^2\ell^2}{R} + f\right)\frac{dx}{dt} + \frac{\ell B}{R}V_{\text{bat}}$$

or

$$m_\ell \frac{d^2x}{dt^2} + \left(\frac{B^2\ell^2}{R} + f\right)\frac{dx}{dt} = \frac{\ell B}{R}V_{\text{bat}}$$

This is the equation of motion for the bar with V_{bat} as the control input.

1.3.2 Back EMF in the Single-Loop Motor

The back-emf induced in the single-loop motor by the external magnetic field of the permanent magnet is now computed. To do so, consider the flux surface shown in Figure 1.14. That is, the surface is a half-cylinder of radius $\ell_2/2$ and length ℓ_1 with the current loop as its boundary. The cylindrical surface is in the air gap, where the magnetic field is known to be radially directed and constant in magnitude, i.e.,

$$\begin{aligned}\vec{\mathbf{B}} &= +B\hat{\mathbf{r}} \text{ for } 0 < \theta < \pi \\ &= -B\hat{\mathbf{r}} \text{ for } \pi < \theta < 2\pi\end{aligned}$$

On the cylindrical part of the surface, the surface element is chosen as

$$d\vec{\mathbf{S}} = (\ell_2/2)d\theta dz\hat{\mathbf{r}}$$

which is directed outward from the axis of the cylinder. The corresponding direction of positive travel is shown in Figure 1.14. On the two ends (half-disks) of the cylindrical surface, the $\vec{\mathbf{B}}$ field is quite weak making the flux through these two half-disks negligible.

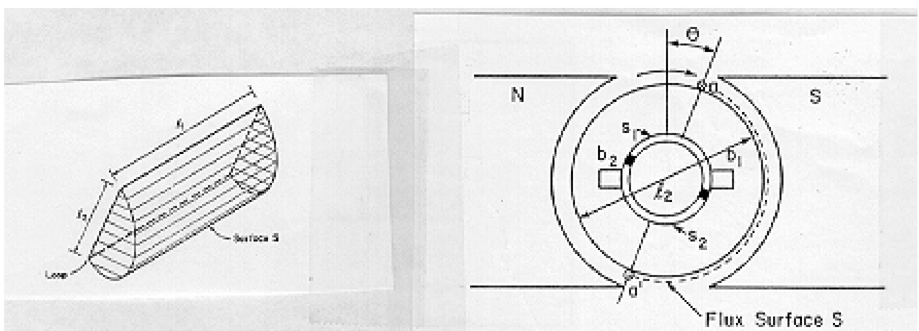


FIGURE 1.14. Flux surface for the single-loop motor. The positive direction of travel around this surface is indicated by the curved arrow.

Thus, neglecting the negligible flux through the two ends of the surface,

$$\begin{aligned}\phi\left(\theta_R\right) &= \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \\ &= \int_0^{\ell_1}\int_{\theta=\theta_R}^{\theta=\pi}(B\hat{\mathbf{r}})\cdot(\frac{\ell_2}{2}d\theta dz\hat{\mathbf{r}}) + \int_0^{\ell_1}\int_{\theta=\pi}^{\theta=\pi+\theta_R}(-B\hat{\mathbf{r}})\cdot(\frac{\ell_2}{2}d\theta dz\hat{\mathbf{r}}) \\ &= \int_0^{\ell_1}\int_{\theta=\theta_R}^{\theta=\pi} B\frac{\ell_2}{2}d\theta dz + \int_0^{\ell_1}\int_{\theta=\pi}^{\theta=\pi+\theta_R} -B\frac{\ell_2}{2}d\theta dz \\ &= \frac{\ell_1\ell_2 B}{2}(\pi-\theta_R) - \frac{\ell_1\ell_2 B}{2}\theta_R, \qquad 0<\theta_R<\pi \\ &= -\ell_1\ell_2 B\left(\theta_R - \frac{\pi}{2}\right), \qquad 0<\theta_R<\pi \qquad (1.1)\end{aligned}$$

As seen from Figure 1.14, this derivation is based on the fact that the $\vec{\mathbf{B}}$ field is directed radially outward over the length $(\ell_2/2)(\pi-\theta_R)$ and radially inward over the length $(\ell_2/2)\theta_R$.

Similarly, it follows that $\phi(\theta_R) = -\ell_1\ell_2 B\left(\theta_R - \pi - \frac{\pi}{2}\right)$ for $\pi < \theta_R < 2\pi$. In general,

$$\phi(\theta_R) = -\ell_1\ell_2 B\left(\theta_R \bmod \pi - \frac{\pi}{2}\right)$$

is the correct expression for any angle θ_R.

Then by (1.1), the induced emf in the rotor loop is

$$\begin{aligned}\xi &= -\frac{d\phi}{dt} \\ &= (\ell_1\ell_2 B)\frac{d\theta_R}{dt} \\ &= K_b\omega_R\end{aligned}$$

where $K_b \triangleq \ell_1\ell_2 B$ is called the back-emf constant.

The total emf in the rotor loop due to the voltage source v_a and external magentic field is $v_a - K_b\omega_R$. How does one know to subtract ξ from the applied voltage v_a? As seen in Figure 1.14, the positive direction of travel around the loop is in *opposition* to v_a, so that if $\xi > 0$, it is opposing the applied voltage v_a. The standard terminology is to call $\xi \triangleq K_b\omega_R$ the "back-emf" of the motor.

1.3.3 Self Induced Emf in the Single Loop Motor

The computation of the flux in the rotor loop produced by its own (armature) *current* is now done. To do so, consider the flux surface shown in Figure 1.15 below.

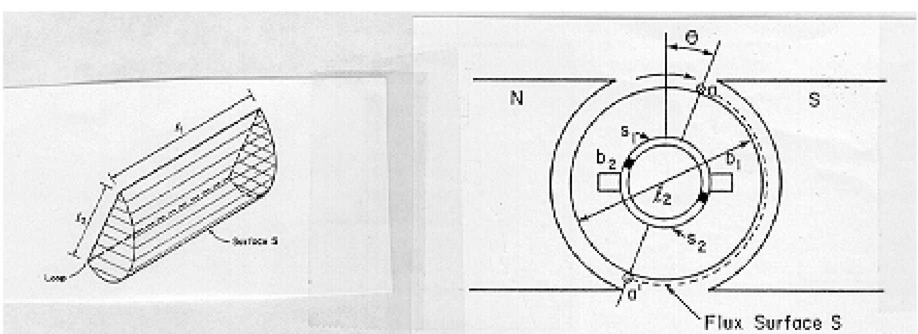


FIGURE 1.15. Computation of the self inductance of the rotor loop. The surface element vector is $d\vec{\mathbf{S}} = -r_R d\theta dz\hat{\mathbf{r}}$ with a resulting positive direction of travel as indicated by the curved arrow. This direction coincides with positive armature current flow, i.e., $i > 0$.

With reference to Figure 1.15, note that the magnetic field on the flux surface *due to the armature current* has the form

$$\vec{\mathbf{B}}(r_R,\theta,i) = K(r_R,\theta)i\,(-\hat{\mathbf{r}})$$

where

$$\begin{aligned} K(r_R,\theta) &> 0 \text{ for } \theta_R \le \theta \le \theta_R + \pi \\ K(r_R,\theta) &< 0 \text{ for } \theta_R + 2\pi \le \theta \le \theta_R + \pi \end{aligned}$$

The exact expression for $K(r_R,\theta)$ is not important for the analysis here. Rather, the point is that with $i > 0$, the magnetic field $\vec{\mathbf{B}}(r_R,\theta,i)$ due to the current in the rotor loop is radially in on the flux surface shown in Figure 1.15, i.e., for $\theta_R \le \theta \le \theta_R + \pi$. For convenience, the surface element is chosen to be $d\vec{\mathbf{S}} = r_R d\theta dz(-\hat{\mathbf{r}})$ so that positive direction of travel around the surface coincides with the positive direction of the armature current i in the loop. The flux[4] ψ in the rotor loop is then computed as

$$\begin{aligned} \psi(i) &= \int_S \vec{\mathbf{B}}\cdot d\vec{\mathbf{S}} = \int_{\theta_R}^{\theta_R+\pi}\int_0^{\ell_1} K(r_R,\theta)i\,(-\hat{\mathbf{r}})\cdot(-r_R d\theta dz\hat{\mathbf{r}}) \\ &= i\int_{\theta_R-\pi}^{\theta_R}\int_0^{\ell_1} K(r_R,\theta)r_R d\theta dz \\ &= Li \text{ where } L = \int_{\theta_R}^{\theta_R+\pi}\int_0^{\ell_1} K(r_R,\theta)r_R d\theta dz > 0. \end{aligned}$$

This last equation just says the flux in the loop (due to the current in the loop) is proportional to the current i in the loop where the proportionality constant L is the called the *self-inductance* of the loop. If $-d\psi/dt = -Ldi/dt > 0$, then the induced emf will force current into the page $\otimes$ on side a and out of the page $\odot$ of side a' in Figure 1.15, That is, this induced emf has the same sign convention as the armature current i and armature voltage v.

If the resistance of the current loop is denoted by R and v is the voltage applied to the loop, then Kirchoff's voltage law leads to

$$v - Ri - Ldi/dt = 0$$

or

$$v = Ri + Ldi/dt$$

The loop and its equivalent circuit are shown below.

[4] The notation ψ is used to distinguish this flux from the flux ϕ in the loop due to the *external* permanent magnet. However, the total flux using an inward normal would be $\psi - \phi$ as the *outward* normal was used to compute ϕ in section 1.3.2.

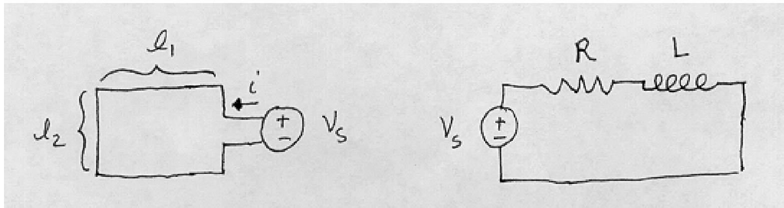


FIGURE 1.16.

The reader should convince him/her self that Lenz's law holds as it must. For example, suppose a voltage $v > 0$ is applied to the loop resulting in both $i > 0$ and $di/dt > 0$, that is, the flux $\phi = Li$ is positive and increasing. The induced voltage is $-d(Li)/dt < 0$ and opposes the current i producing the increasing flux $\phi = Li$. In this circumstance, the voltage source v is forcing the current i against this induced voltage $-d(Li)/dt$ and the power absorbed by the induced voltage is $-id(Li)/dt = -d\left(\frac{1}{2}Li^2\right)/dt$. This power is stored in the energy $\frac{1}{2}Li^2$ of the magnetic field surrounding the loop.

Arcing between the commutator and brushes in a DC machine

Suppose the single-loop motor is rotating at constant speed ω_0 with a constant current i_0 in the rotating loop. Let L be the self-inductance of the loop. Now, every half-turn, the current in the loop reverses direction as shown in Figures 1.8b,c,d. During this commutation, the current goes from i_0 to 0 to $-i_0$ with change in flux in the loop given by $\Delta\psi = Li_0 - L(-i_0) = 2Li_0$. By Faraday's Law, the (self) induced emf is then $-\Delta\psi/\Delta t$ where Δt is the time for the current to change direction. Note that this time Δt decreases as the motor speed increases, so that, even if L is small, the induced emf in the loop (due to the reversal of current in the loop) can be quite large at high motor speeds. These large voltages may cause arcing from the slip rings to the brushes and, if not damage, at least give unwanted transient currents in the armature circuit.

1.4 Dynamic Equations of the DC Motor

Based on the simple single-loop DC motor analyzed above, the complete set of equations for a DC motor can be found. The total emf (voltage) in the loop due to the voltage source v_a, the external permanent magnet and the changing current i in the rotor loop is

$$v_a - K_b\omega_R - Ldi/dt.$$

This voltage goes into building up the current in the loop against the loop's resistance, that is,

$$v_a - K_b\omega_R - Ldi/dt = Ri$$

or

$$L\frac{di}{dt} = -Ri - K_b\omega_R + v_a.$$

This relationship is often illustrated by the equivalent circuit given in Figure 1.17 below.

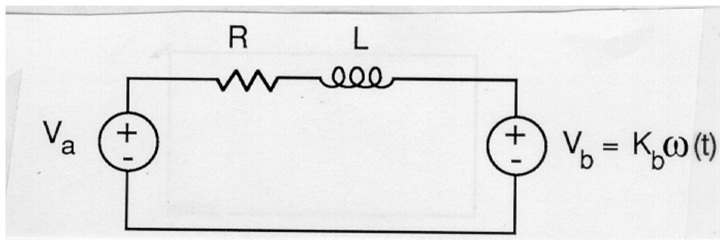


FIGURE 1.17. Equivalent Circuit for the DC Motor

Recall that the torque τ_m on the loop due to the external magnetic field acting on the current in the loop is

$$\tau_m = K_T i$$

where $K_T \triangleq \ell_1\ell_2 B$ is called the torque constant. By connecting a shaft and gears to one end of the loop this motor torque can be used to do work (lift weight etc.). It was also shown that $K_T = K_b = \ell_1\ell_2 B$ and is a consequence of energy conservation. Let f be the coefficient of viscous-friction (i.e., due to the brushes, bearings, etc.) and let τ_L be the load torque (e.g., to lift a weight). Then, by Newton's Law,

$$\tau_m - \tau_L - f\omega_R = Jd\omega_R/dt$$

where J is the moment of inertia of rotor assembly (armature, etc.). The system of equations characterizing the DC motor is then

$$\begin{aligned} Ldi/dt &= -Ri - K_b\omega_R + v_a \\ Jd\omega_R/dt &= K_T i - f\omega_R - \tau_L \\ d\theta_R/dt &= \omega_R \end{aligned}$$

A picture of a DC motor servo system and its associated schematic is shown in Figure 1.18. In the schematic, R, L represent the resistance and inductance, respectively, of the armature loop, $V_b = K_b\omega_R$ denotes the back emf, $\tau_m = K_T i$ denotes the motor torque, and J, f are rotor moment of inertia and viscous friction coefficient, respectively. The positive directions for τ_m, θ_R and τ_L are indicated by the curved arrows. The fact that the curved arrow for τ_L is opposite to that of τ_m just means that if the load torque is positive then it opposes a positive motor torque τ_m. The circuit denoted with i_f = constant represents the field winding and simply means the above equations are valid for a DC motor with an electromagnet assuming the field current of the electromagnet is kept constant (so that radial B field in the airgap is constant). If a permanent magnet is used, then of course the B field in the airgap is constant.

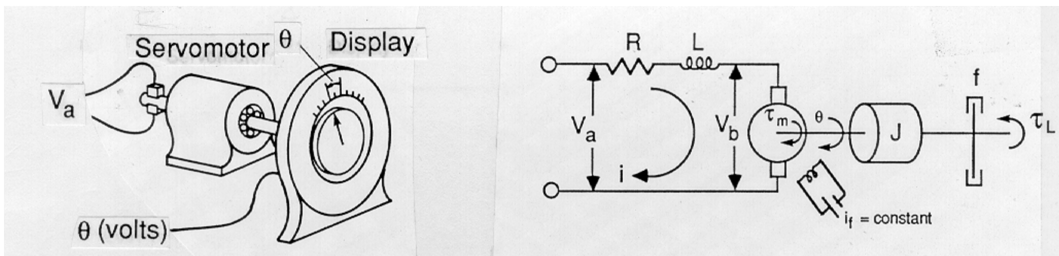


FIGURE 1.18. DC motor picture and schematic

Electromechanical Energy Conversion

The mechanical power produced by this DC motor is $\tau\omega_R = i\ell_1\ell_2 B\omega_R$ while the electrical power absorbed by the back-emf is $i\xi = i\ell_1\ell_2 B\omega_R$. That is, the electrical power absorbed by the back-emf equals (is converted to) the mechanical power produced. Another way to view this energy conversion is to write the rotor electrical equation as

$$v_a = Ri + Ldi/dt + K_b\omega_R$$

The power out of the voltage source $v_a(t)$ is $v_a(t)i(t)$ and

$$\begin{aligned} v_a(t)i(t) &= Ri^2(t) + Lidi/dt + iK_b\omega_R \\ &= Ri^2 + \frac{d}{dt}\frac{1}{2}Li^2 + i\xi \\ &= Ri^2 + \frac{d}{dt}\frac{1}{2}Li^2 + \tau\omega_R \end{aligned}$$

Thus the power $v_a(t)i(t)$ delivered by the source goes into heat loss in the resistance R, into stored magnetic energy in the inductance L of the loop and the amount $i\xi$ goes into the mechanical energy $\tau\omega_R$.

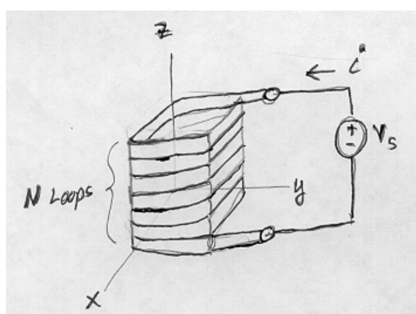
1.5 Flux linkage

Here Faraday's Law is described in the context of flux linkage. To introduce the notion of flux linkage, consider a multi-loop coil as shown in the figure below. This is an air core solenoid made by tightly wrapping wire around an air core. When current is applied to the wire, each loop of the wire has approximately the same flux $\psi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}$ as the $\vec{\mathbf{B}}$ is approximately constant inside the solenoid if the number of turns N is large. That is, $\vec{\mathbf{B}} \approx B_0 i \hat{\mathbf{z}}$ *inside* the solenoid. As a result, each loop of the wire has the same induced voltage in it given by $-d\psi/dt$. Choose the surface element vectors as $d\vec{\mathbf{S}} = dxdy\hat{\mathbf{z}}$ for each loop so that the sign convention for the induced voltages is the same for all loops and the flux is given by $\psi = (B_0 S)\, i = L_1 i$ where S is the cross sectional area of the solenoid and $L_1$is the (self) inductance of each loop. Also, note that choosing $d\vec{\mathbf{S}} = dxdy\hat{\mathbf{z}}$, the sign convention for the induced voltage $-d\psi/dt$ in each loop coincides with that of the current i and applied voltage v_s. The total induced voltage in the length of the wire is just the sum of the voltages induced in each loop, that is, $v_{\text{solenoid}} = -Nd\psi/dt = -d\,(N\psi)\,dt$. Define the *flux linkage* as $\lambda \triangleq N\psi$ which is simply the sum of the fluxes in all the loops making up the coil. Then $v_{\text{solenoid}} = -d\lambda/dt$. It is almost always more convenient to compute the flux linkage when computing the total induced voltage in a wire which has many turns (loops). Using Kirchoff's law, the equation for the loop is then

$$-\frac{d}{dt}\lambda + v_s - Ri = 0$$

Or, with $L = nL_1$ the total inductance of the loop and R the total resistance of the loop, this becomes

$$L\frac{di}{dt} = -Ri + v_s$$



Example Distributed windings

To show how flux linkage is used in AC machines, consider Figure 1.19 which shows four loops wound in the inside surface of cylinder of soft iron material. That is, a single piece of wire is wound around the inside surface where one loop is placed in a slot at $\theta = \pi/3$ (the other side of the loop is in the slot at $\theta = \pi/3 - \pi$) then two loops are wound at $\theta = \pi/2$ and finally one loop is wound at $\theta = 2\pi/3$. One end of the wire is labeled a and the other a' and the objective here is to calculate the total emf $\xi_{a-a'}$ induced in the wire by the permanent magnet rotor.

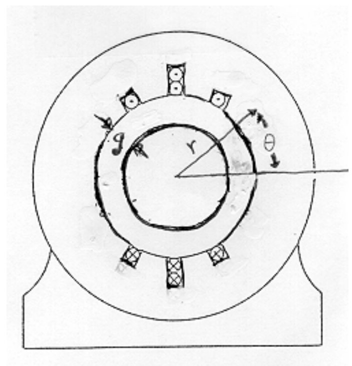


FIGURE 1.19. Phase a winding.

As before, if the current in the stator winding is positive, the symbol $\odot$ means it is coming out of the page and the symbol $\otimes$ means it is going into the page. Here it is assumed that the permanent magnet produces a magnetic field in the airgap given by

$$\vec{\mathbf{B}}_R(\theta - \theta_R) = B_{\max}\frac{r_R}{r}\cos(\theta - \theta_R)\hat{\mathbf{r}}.$$

where r_R is the radius of the rotor, (r, θ) are the polar coordinates of an arbitrary location in the airgap and θ_R is the rotor angle defined by the centerline of the rotor's north pole. This rotor magnet produces a flux in each loop and for any given rotor position θ_R, this flux is different in the loop at $\theta = \pi/3$, the loops at $\pi/2$ and the loop at $2\pi/3$. Further, if the rotor is moving, then it is producing a changing flux in each of the four loops and therefore, by Faraday's law, this changing flux will produce an emf in each of the four loops. Of course, the emfs will be different in the loop at $\theta = \pi/3$, the loops at $\pi/2$ and the loop at $2\pi/3$.

To compute these emfs, let

$$d\vec{\mathbf{S}} = r_S d\theta d\ell\hat{\mathbf{r}}$$

where r_S is the radius of the inside surface of the stator iron. Note that with this choice of $d\vec{\mathbf{S}}$, the positive direction of travel around the loop coincides with the positive direction of current in that loop (see Figure 1.19). On the inside surface of the rotor, $r = r_S$ so that the flux in the loop whose sides are in the slots at $\theta = \pi/3$ and $\theta = \pi/3 - \pi$ is

$$\begin{aligned}
\phi_{\pi/3} &= \int_{\substack{\text{Loop from} \\ \pi/3-\pi \text{ to } \pi/3}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} \\
&= \int_0^{\ell_1} \int_{\theta=\pi/3-\pi}^{\theta=\pi/3} B_{\max} \frac{r_R}{r_S} \cos(\theta - \theta_R)\hat{\mathbf{r}} \cdot (r_S d\theta d\ell \hat{\mathbf{r}}) \\
&= \ell_1 r_R \int_{\theta=\pi/3-\pi}^{\theta=\pi/3} B_{\max} \cos(\theta - \theta_R) d\theta \\
&= \ell_1 r_R B_{\max} \sin(\theta - \theta_R) d\theta \Big|_{\theta=\pi/3-\pi}^{\theta=\pi/3} \\
&= 2\ell_1 r_R B_{\max} \sin(\pi/3 - \theta_R)
\end{aligned}$$

Then the emf induced in this loop by the magnetic field of the permanent magnet is

$$\xi_{\pi/3} = -\frac{d\phi_{\pi/3}}{dt} = 2\ell_1 r_R B_{\max} \omega_R \cos(\theta_R - \pi/3)$$

where $\omega_R \triangleq d\theta_R/dt$. If $\xi_{\pi/3} > 0$, this emf will force current to go in the positive direction of travel around the loop which coincides with the positive direction of current in that loop.

Using the same $d\vec{\mathbf{S}}$ as before, the flux in each of the two loops between $-\pi/2$ to $\pi/2$ is

$$\phi_{\pi/2} = \int_{\substack{\text{Loop from} \\ -\pi/2 \text{ to } \pi/2}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = 2\ell_1 r_R B_{\max} \sin(\pi/2 - \theta_R)$$

and the induced emf in each of the loops is

$$\xi_{\pi/2} = -\frac{d\phi_{\pi/2}}{dt} = 2\ell_1 r_R B_{\max} \omega_R \cos(\theta_R - \pi/2).$$

If $\xi_{\pi/2} > 0$, it will force current to go in the positive direction of travel around the loop which also coincides with the positive direction of current in that loop.

Finally,

$$\phi_{2\pi/3} = \int_{\substack{\text{Loop from} \\ 2\pi/3-\pi \text{ to } 2\pi/3}} \vec{\mathbf{B}}_R \cdot d\vec{\mathbf{S}} = 2\ell_1 r_R B_{\max} \sin(2\pi/3 - \theta_R)$$

and

$$\xi_{2\pi/3} = -\frac{d\phi_{2\pi/3}}{dt} = 2\ell_1 r_R B_{\max} \omega_R \cos(\theta_R - 2\pi/3).$$

Again, if $\xi_{2\pi/3} > 0$, it will force current to go in the positive direction of travel around the loop coinciding with the positive direction of current in that loop.

All four loops were chosen to have the same sign convention for positive travel around the loop and each coincides with the positive direction of current in that loop. The induced emfs in the loops are all in series in the winding and, as they all have the *same* sign convention, they can be added up to get the total emf in the phase winding. That is,

$$\begin{aligned}\xi_{a-a'} &= \xi_{\pi/3} + 2\xi_{\pi/2} + \xi_{2\pi/3} \\ &= -\left(\frac{d\phi_{\pi/3}}{dt} + 2\frac{d\phi_{\pi/2}}{dt} + \frac{d\phi_{2\pi/3}}{dt}\right) \\ &= -\frac{d}{dt}\left(\phi_{\pi/3} + 2\phi_{\pi/2} + \phi_{2\pi/3}\right) \\ &= -\frac{d}{dt}\lambda_{a-a'}\end{aligned}$$

where

$$\lambda_{a-a'} \triangleq \phi_{\pi/3} + 2\phi_{\pi/2} + \phi_{2\pi/3}$$

is the total *flux linkage* in phase $a - a'$. The point here is that one can first sum the fluxes in all the loops of a phase winding (i.e., compute the flux linkage) and then apply Faraday's law to the resulting flux linkage to get the total emf in the phase. However, as done above, care must be taken to ensure that the flux in each loop is computed in a consistent fashion so that the resulting emfs all have the same sign convention and therefore add up to give the total emf in the phase winding.

In the case where the rotor is moving at constant angular speed with $\theta_R = \omega_R t$ and $V_{\max} \triangleq 2\ell_1 r_R B_{\max}\omega_R$, the total induced emf $\xi_{a-a'}$ in the phase $a - a'$ is then

$$\begin{aligned}\xi_{a-a'} &= V_{\max}\operatorname{Re}\left\{e^{j(\omega_R t-\pi/3)} + 2e^{j(\omega_R t-\pi/2)} + e^{j(\omega_R t-2\pi/3)}\right\} \\ &= V_{\max}\operatorname{Re}\left\{e^{j\omega_R t}\left(e^{-j\pi/3} + 2e^{-j\pi/2} + e^{-j2\pi/3}\right)\right\} \\ &= V_{\max}(2+\sqrt{3})\cos(\omega_R t - \pi/2)\end{aligned}$$

as $e^{-j\pi/3} + 2e^{-j\pi/2} + e^{-j2\pi/3} = (2 + 2\cos(\pi/6)e^{-j\pi/2} = (2+\sqrt{3})e^{-j\pi/2}$

1.6 Microscopic Viewpoint of the Back Emf and Torque in DC Machines

Additional insight into the (back) emf ξ is found by calculating it from a microscopic point of view, using the ideas given in [12] (see P. 887). Recall that the magnetic force on a charged particle q is $\vec{\mathbf{F}}_{\text{magnetic}} = q\vec{\mathbf{v}}\times\vec{\mathbf{B}}$, where $\vec{\mathbf{v}}$ is the velocity of the charge (see [12] P. 816). To illustrate this approach, the back emf in the linear DC machine is recomputed from the microscopic point of view.

Example *A Linear DC Machine*

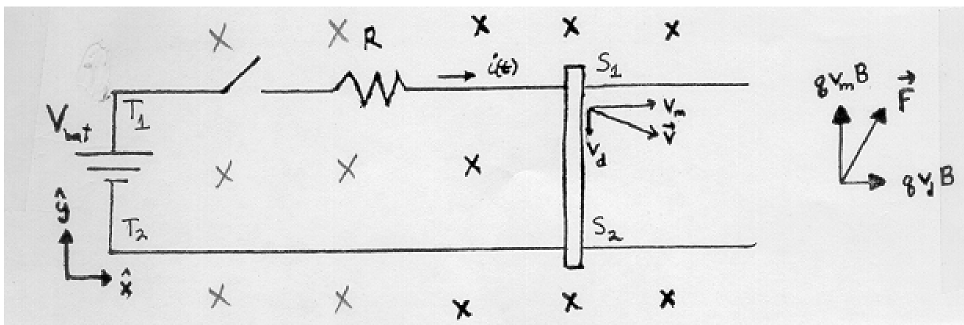


FIGURE 1.20. Linear DC machine

Suppose the motor (bar) is moving to the right with a constant speed v_m. Each charge q in the sliding bar has total velocity $\vec{\mathbf{v}} = v_m\hat{\mathbf{x}} - v_d\hat{\mathbf{y}}$, where v_d is the drift speed of the charges down the wire. The magnetic force on the charge q is

$$\begin{aligned}\vec{\mathbf{F}}_{\text{magnetic}} &= q\vec{\mathbf{v}}\times\vec{\mathbf{B}}\\ &= q(v_m\hat{\mathbf{x}} - v_d\hat{\mathbf{y}})\times(-B\hat{\mathbf{z}})\\ &= qv_mB\hat{\mathbf{y}} + qv_dB\hat{\mathbf{x}}\end{aligned}$$

Now, the component of force $qv_dB\hat{\mathbf{x}}$ perpendicular to the bar causes the bar to move to the right and the component $qv_mB\hat{\mathbf{y}}$ along the bar opposes the current flow. The battery sets up an electric field $\vec{\mathbf{E}}_{\text{battery}}$ in the bar to overcome the magnetic force $qv_mB\hat{\mathbf{y}}$ so as to make the current flow (setup the drift velocity v_d against the resistance of the bar). In more detail, with T_1 and T_2 the upper and lower terminals of the battery respectively, and S_1 and S_2 the upper and lower sliding contact points, respectively, the battery voltage is given as

$$V_{\text{battery}} = \int_{T_1-S_1-S_2-T_2} \vec{\mathbf{E}}_{\text{battery}}\cdot d\vec{\ell}$$

$q\vec{\mathbf{E}}_{\text{battery}}$ is the force on each charge carrier and $qV_{\text{battery}} = \int_{T_1-S_1-S_2-T_2} q\vec{\mathbf{E}}_{\text{battery}} \cdot d\vec{\ell}$ is the energy given to the charge carrier by the battery source as the charge goes around the loop. There is also a magnetic force on the charge carrier that opposes the electric field $\vec{\mathbf{E}}_{\text{battery}}$. The energy per unit charge ξ that the magnetic force takes from the charge carrier as it goes down the bar from S_1 to S_2 is given by

$$\begin{aligned}
\xi &= \frac{1}{q}\int_{S_1}^{S_2} \vec{\mathbf{F}}_{\text{magnetic}} \cdot d\vec{\ell} \\
&= \frac{1}{q}\int_{S_1}^{S_2} q(\vec{\mathbf{v}}\times\vec{\mathbf{B}}) \cdot d\vec{\ell} \\
&= \int_0^{\ell} (v_d B\hat{\mathbf{x}}+v_m B\hat{\mathbf{y}}) \cdot (-d\ell\hat{\mathbf{y}}) \\
&= -v_m B\ell
\end{aligned}$$

The fact that ξ is *negative* just indicates that the magnetic force is taking energy out of the charge carrier as it goes down the bar from S_1 to S_2. This energy/unit-charge ξ given to (taken from) each charge carrier by the magnetic-force as it goes around the loop is called the induced emf. The voltage V_{battery} was computed by integrating $\vec{\mathbf{E}}_{\text{battery}}$ in the *clockwise* direction $T_1 - S_1 - S_2 - T_2$ around the loop and ξ by integrating $\vec{\mathbf{v}}\times\vec{\mathbf{B}}$ also in the *clockwise* direction $S_1 - S_2$ down the bar, that is, they both have the same sign convention. This is in contrast to the macroscopic case where V_{battery} and ξ had opposite sign conventions resulting in ξ $(= v_m B\ell)$ being positive. However, the same (physical) result occurs as in the macroscopic case.

The total emf is the sum of the battery voltage and induced emf and this total emf goes into producing the current, i.e.,

$$V_{\text{battery}} + \xi = V_{\text{battery}} - v_m \ell B = Ri$$

where an identical equation was found in the macroscopic case using Faraday's Law. Let $\vec{\mathbf{F}}_\ell$ denote the total magnetic-force on the bar in the $\hat{\mathbf{x}}$ direction. Then,

$$\vec{\mathbf{F}}_\ell \triangleq q(NS\ell)v_d B\hat{\mathbf{x}}$$

where N is the number of charge carriers/volume and S is the cross-sectional area of the sliding bar. That is, $NS\ell$ is the total number of charge carriers in the sliding bar each experiencing the force $qv_d B\hat{\mathbf{x}}$. As illustrated in Figure 1.21, in a time Δt, the charges in the volume $NS(v_d\Delta t)$ have moved down the bar past the point A in Figure 1.21.

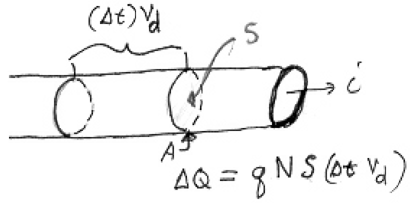


FIGURE 1.21. In the time Δt, the amount of charge $\Delta Q = qNS(v_d \Delta t)$ has moved past the point A resulting in the current $i = \Delta Q/\Delta t = qNSv_d$ in the bar.

That is, the amount of charge $\Delta Q = qNS(v_d \Delta t)$ has moved past the point A in the time Δt resulting in the current $i = \Delta Q/\Delta t = qNSv_d$ in the bar. Consequently, the total magnetic force on bar may be rewritten as

$$\begin{aligned} \vec{\mathbf{F}}_\ell &\triangleq q(NS\ell)v_d B\hat{\mathbf{x}} \\ &= (qNSv_d)\ell B\hat{\mathbf{x}} \\ &= i\ell B\hat{\mathbf{x}} \end{aligned}$$

which is identical to the expression derived from the macroscopic point of view.

1.6.1 Application to the Single-Loop DC Motor

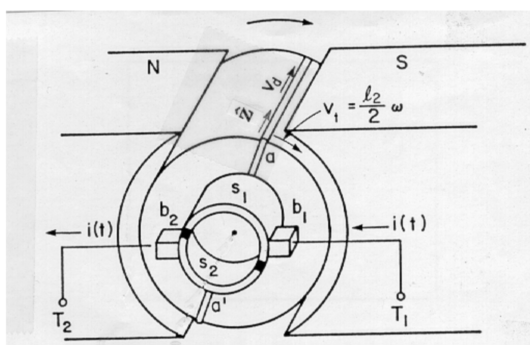


FIGURE 1.22. Single-loop DC motor

When the loop is rotating at angular speed ω_R, the velocity of the charge carriers that make up the current is given by

$$\begin{aligned} \vec{\mathbf{v}} &= v_t\hat{\boldsymbol{\theta}} + v_d\hat{\mathbf{z}} \text{ for side } a \\ &= v_t\hat{\boldsymbol{\theta}} - v_d\hat{\mathbf{z}} \text{ for side } a' \end{aligned}$$

where v_d is the drift speed of the charge carriers along the wire and $v_t = (\ell_2/2)\omega_R$ is the tangential velocity due to the rotating loop. Recall that the drift speed has the same sign as the current, that is, $v_d > 0$ for $i > 0$. Also recall that the angular velocity is written as $\vec{\boldsymbol{\omega}}_R = \omega_R \hat{\mathbf{z}}$ since $\hat{\mathbf{z}}$ is the axis the motor is turning about. The magnetic-force/unit-charge $\vec{\mathbf{F}}_{\text{magnetic}}/q$ on the charge carriers on the axial sides of the loop is

$$\vec{\mathbf{F}}_{\text{magnetic}}/q = \vec{\mathbf{v}}\times\vec{\mathbf{B}}$$

where

$$\begin{aligned}\vec{\mathbf{v}}\times\vec{\mathbf{B}} &= (v_t\hat{\boldsymbol{\theta}}+v_d\hat{\mathbf{z}}) \times (+B)\hat{\mathbf{r}} = -v_t B\hat{\mathbf{z}}+v_d B\hat{\boldsymbol{\theta}} = v_d B\hat{\boldsymbol{\theta}} - \omega_R(\ell_2/2)B\hat{\mathbf{z}} \text{ on side } a \\ &= (v_t\hat{\boldsymbol{\theta}}-v_d\hat{\mathbf{z}}) \times (-B)\hat{\mathbf{r}} = +v_t B\hat{\mathbf{z}}+v_d B\hat{\boldsymbol{\theta}} = v_d B\hat{\boldsymbol{\theta}} + \omega_R(\ell_2/2)B\hat{\mathbf{z}} \text{ on side } a'\end{aligned}$$

Now, the $v_d B\hat{\boldsymbol{\theta}}$ term is what produces the torque. In more detail, with N the number of charge carriers/unit-volume, S the cross-sectional area of the wire loop and ℓ_1 the axial length of the loop, then $NS\ell_1$ is the total number of charge carriers of each side of the loop and the current is then given by $i = qNSv_d$ (see Figure 1.21). The total tangential forces on the axial sides of the rotor loop are given by

$$\begin{aligned}\vec{\mathbf{F}}_{\text{side } a} &= (qNS\ell_1)v_d B\hat{\boldsymbol{\theta}} = i(t)\ell_1 B\hat{\boldsymbol{\theta}} \\ \vec{\mathbf{F}}_{\text{side } a'} &= (qNS\ell_1)v_d B\hat{\boldsymbol{\theta}} = i(t)\ell_1 B\hat{\boldsymbol{\theta}}\end{aligned}$$

The torque is then

$$\begin{aligned}\vec{\boldsymbol{\tau}} &= \frac{\ell_2}{2}\hat{\mathbf{r}}\times\vec{\mathbf{F}}_{\text{side } a} + \frac{\ell_2}{2}\hat{\mathbf{r}}\times\vec{\mathbf{F}}_{\text{side } a'} \\ &= 2(\frac{\ell_2}{2}\hat{\mathbf{r}}) \times (i\ell_1 B\hat{\boldsymbol{\theta}}) \\ &= i\ell_1\ell_2 B\hat{\mathbf{z}}\end{aligned}$$

which is the same result as in the macroscopic case.

It is now shown that the $\hat{\mathbf{z}}-$component of the magnetic force produces the back emf. The $\hat{\mathbf{z}}-$component of $\vec{\mathbf{F}}_{\text{magnetic}}$ (i.e., along the axial sides of the loop) is given by

$$\begin{aligned}(\vec{\mathbf{F}}_{\text{magnetic}}/q)_z\hat{\mathbf{z}} &= -\omega_R(\ell_2/2)B\hat{\mathbf{z}} \text{ for side } a \\ &= +\omega_R(\ell_2/2)B\hat{\mathbf{z}} \text{ for side } a'\end{aligned}$$

As seen in Figure 1.22, this component of the magnetic-force per unit-charge $(\vec{\mathbf{F}}_{\text{magnetic}}/q)_z$ opposes the electric field $\vec{\mathbf{E}}_a$ set up in the loop by the applied armature voltage v_a. In more detail,

$$v_a = \int_{T_1}^{T_2} \vec{\mathbf{E}}_a \cdot d\vec{\ell}$$

where

$$\begin{aligned} d\vec{\ell} &= +d\ell\ \hat{\mathbf{z}} \text{ for side } a \\ &= -d\ell\ \hat{\mathbf{z}} \text{ for side } a' \end{aligned}$$

while the back-emf is given by

$$\begin{aligned} \xi &\triangleq \int_{T_1}^{T_2} (\vec{\mathbf{F}}_{\text{magnetic}}/q) \cdot d\vec{\ell} \\ &= \int_{\text{side } a} (-\omega_R(\ell_2/2)B\hat{\mathbf{z}}) \cdot (d\ell\hat{\mathbf{z}}) + \int_{\text{side } a'} (\omega_R(\ell_2/2)B\hat{\mathbf{z}}) \cdot (-d\ell\hat{\mathbf{z}}) \\ &= \int_{\ell=0}^{\ell=\ell_1} -\omega_R(\ell_2/2)Bd\ell + \int_{\ell=0}^{\ell=\ell_1} -\omega_R(\ell_2/2)Bd\ell \\ &= -\omega_R(\ell_2/2)B\ell_1 - \omega_R(\ell_2/2)B\ell_1 \\ &= -\ell_1\ell_2 B\omega_R \end{aligned}$$

The minus sign just indicates that ξ opposes the applied armature voltage v_a. The total emf in the loop is

$$v_a + \xi = v_a - \ell_1\ell_2 B\omega_R.$$

Finally then, the equation governing the current in the rotor loop is

$$L\frac{di}{dt} + Ri = v_a - \ell_1\ell_2 B\omega_R.$$

This is the same physical result as shown in the macroscopic case using Faraday's Law. However, here the induced emf $\xi = -\ell_1\ell_2 B\omega_R$ is negative because it was chosen to have the same sign convention as v_a (i.e., both v_a and ξ are positive going from T_1 to T_2). This is in contrast to the macroscopic case in which they had opposite sign conventions so that $\xi = \ell_1\ell_2 B\omega_R$ was positive, but still opposed v_a.

Remark *Voltage and Emf*

The *electromotive force* or *emf* between two points in a circuit is the integral of the total force per unit-charge along the circuit between those two points[5]. This force per unit-charge can be due to an electric field, a magnetic field or both. As seen above, the emf in motors is typically due to both. The term voltage (drop) is usually reserved for the integral of the *electric field* between the two points. However, this distinction is usually not made and the two terms (voltage and emf) are used interchangeably.

1.6.2 Drift Speed

Above, it was shown that the drift speed of the charge carriers making up the current is given by $v_d(t) = i(t)/(qNS)$. As explained in [12] (see p.781),

[5] Note that the electromotive force is *not* a force, but rather an energy per unit charge.

this motion (drift speed) of the charges in the conductor is caused by the electric field setup in the conductor by the voltage source and/or induced emfs in the conducting circuit. This electric field is pushing the charges along the conductor against the internal resistance of the conductor. In metals, the outer valence electrons are free to move about the lattice of the metal and are called *conduction electrons*. For example, in copper there is one valence electron per atom and the other 28 electrons remain bound to the copper nucleus. Consequently, as there are 8.4×10^{22} atoms/cm^3 in copper, there are $N = 8.4 \times 10^{22}$ electrons/cm^3 that can move freely within the copper lattice to make up the current in the wire. To consider a simple numerical example, let $i = 10$ Amp, $S = 0.1$ cm^2 so that with $q = 1.6 \times 10^{-19}$ coulomb/electron, the corresponding drift speed is

$$\begin{aligned} v_d &= \frac{10 \text{ Amp}}{(1.6 \times 10^{-19} \text{ coulomb/electron})(8.4 \times 10^{22} \text{ electrons/cm}^3)(.1\text{cm}^2)} \\ &= .74\frac{\text{cm}}{\text{sec}} \end{aligned}$$

That is, it takes $1/(.74\frac{\text{cm}}{\text{sec}}) = 1.35$ seconds for the charge carrier to travel one centimeter. However, it should be noted that that when a voltage and/or emf is applied to a circuit, the corresponding electric field is setup around the circuit at a speed close to the speed of light. This is analogous to applying pressure to a long tube of water. The pressure wave is transmitted down the tube rapidly (at the speed of sound in water) while the water itself moves much slower [12].

1.7 Speed Sensor-Tachometer for a DC Machine

A tachometer is a device for measuring the speed of a DC motor by putting out a voltage proportional to the motor's speed. Consider now a tachometer for the simple linear DC machine.

Example *A Tachometer for a Linear DC Machine*

Consider the figure below, where a device for measuring speed (tachometer) has been added to the linear DC machine.

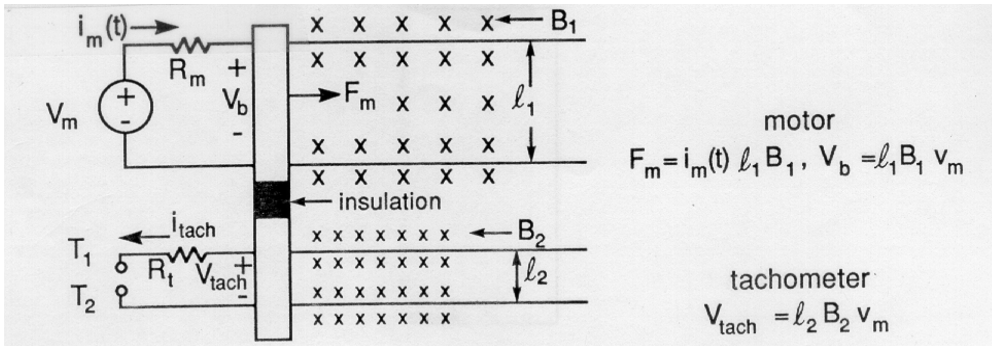


FIGURE 1.23. DC tachometer (generator)

The two bars are rigidly connected together by the insulating material. The motor force (the magnetic force on the upper bar) is $F_{\text{motor}} = i\ell_1 B_1$, and the induced emf in the motor is $\xi = V_b = B_1\ell_1 v$, where v is the speed of the motor (bar).

The induced (back) emf in the tachometer is given by $\xi = v_{tach} = B_2\ell_2 v$ so that by measuring the voltage between the terminals T_1 & T_2, the speed v of the motor can be computed. Note that the tachometer and motor have the same physical structure. In fact, the tachometer is nothing more than a generator putting out a voltage proportional to the speed.

Normally, the output of the tachometer is put through an amplifier, whose output is then fed back to the motor as shown in Figure 1.24.

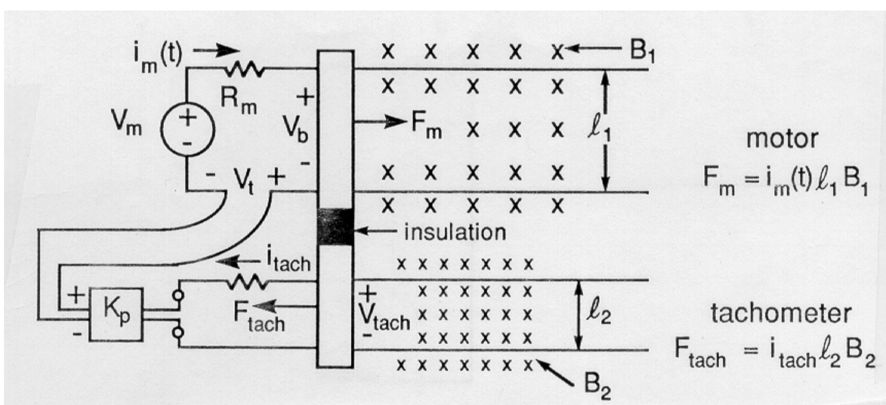


FIGURE 1.24. Tachometer used for speed feedback

With R_{in} the input impedance of the amplifier, the equivalent circuit schematic for the tachometer is as shown.

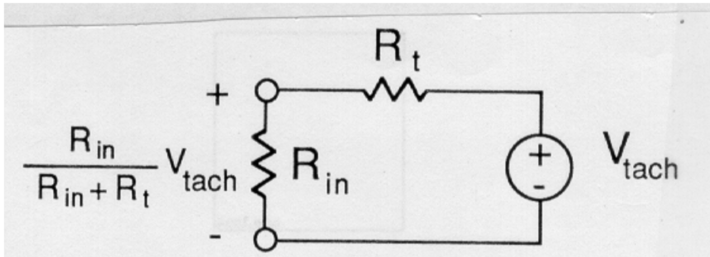


FIGURE 1.25. Circuit schematic for a tachometer

Let v_t be the voltage fed back to the motor as shown above. Then

$$\begin{aligned} v_t &= K_p R_{in}/(R_t + R_{in}) V_{\text{tach}} \\ &\triangleq K_t V_{\text{tach}} \end{aligned}$$

The current is $i_{\text{tach}}(t) = v_{\text{tach}}/(R_t + R_{in})$ which is small if $R_{in} + R_t$ is large. Why must the current be small? Well, the

magnetic force on the tachometer bar is $F_{\text{tach}} = i_{\text{tach}}(t)\ell_2 B_2$ which *opposes* the motion of the bar. Thus the total force F on the two rigidly connected bars is $F = i_{\text{motor}}(t)\ell_1 B_1 - i_{\text{tach}}(t)\ell_2 B_2 - (f_m + f_t)v$ where f_m and f_t are the viscous coefficients of friction of the motor and tachometer, respectively. Clearly, by keeping the tachometer current $i_{\text{tach}}(t)$ small, the opposing force F_{tach} is small. Also, note that

$$\begin{aligned} V_{\text{tach}}(t) i_{\text{tach}} &= \ell_2 B_2 v(t) i_{\text{tach}}(t) \\ &= F_{\text{tach}}(t) v(t) \end{aligned}$$

That is, the mechanical power $F_{\text{tach}}(t)v(t)$ absorbed by the tachometer bar reappears as the electrical power $V_{\text{tach}}(t)i_{\text{tach}}$.

1.7.1 Tachometer for the Single Loop DC Motor

A tachometer for the single-loop DC motor is constructed by attaching another loop to the shaft and rotating it an external magnetic field to act as a DC generator. That is, the changing flux in the loop produces (generates) an induced emf according to Faraday's law and this emf is proportional to the shaft's speed. To see this, consider Figure 1.26, where a motor loop is driven by a voltage v_a and, attached to the same shaft, is a second loop called a tachometer. Both loops rotate in an external radial magnetic field which is not shown in Figure 1.26, but is shown for the tachometer loop in Figure 1.27. It is important to point out that no voltage is applied to the terminals T_1, T_2 of the tachometer as was the case for the motor. Instead, the terminals T_1, T_2 of the tachometer are connected to an amplifier with gain K_a. The voltage V_{tach} put out by the tachometer is proportional to the motor speed ω. To show this, denote the tachometer loop's self-inductance as L_t and its resistance as R_t.

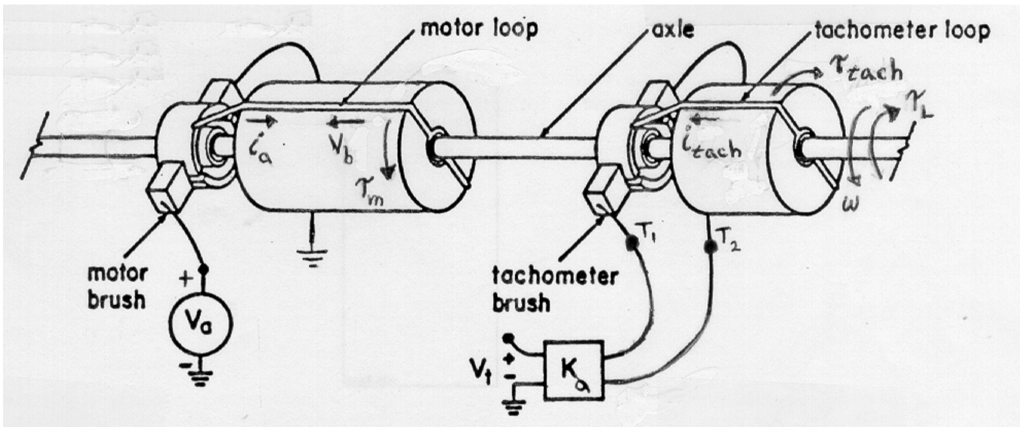


FIGURE 1.26. Single-loop motor and tachometer

In the same way the back emf was computed for the DC motor, one can calculate the flux in the loop of the tachometer due to the external magnetic field. Specifically, (see Figure 1.27):

$$\begin{aligned}
\phi &= \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} \\
&= \int_0^{\ell_1} \int_{\theta_R}^{\pi} (B\hat{\mathbf{r}}) \cdot (\frac{\ell_2}{2} d\theta dz \hat{\mathbf{r}}) + \int_0^{\ell_1} \int_{\pi}^{\pi+\theta_R} (-B\hat{\mathbf{r}}) \cdot (\frac{\ell_2}{2} d\theta dz \hat{\mathbf{r}}) \\
&= \int_0^{\ell_1} \int_{\theta_R}^{\pi} B\frac{\ell_2}{2} d\theta dz + \int_0^{\ell_1} \int_{\pi}^{\pi+\theta_R} -B\frac{\ell_2}{2} d\theta dz \\
&= (\ell_1 \ell_2 B/2)(\pi - \theta_R) - (\ell_1 \ell_2 B/2)\theta_R \\
&= -\ell_1 \ell_2 B \theta_R + (\ell_1 \ell_2 B/2)\pi
\end{aligned}$$

The induced emf is then

$$\begin{aligned}
v_{\text{tach}} &= -d\phi/dt \\
&= (\ell_1 \ell_2 B) d\theta_R/dt \\
&= K_{b_\text{tach}} \omega_R
\end{aligned}$$

where $K_{b\_\text{tach}} = \ell_1 \ell_2 B$ is a constant depending on the dimensions of the tachometer rotor and the strength of the external magnetic field of the tachometer. This shows that the voltage between the terminals T_1, T_2 is proportional to the angular speed.

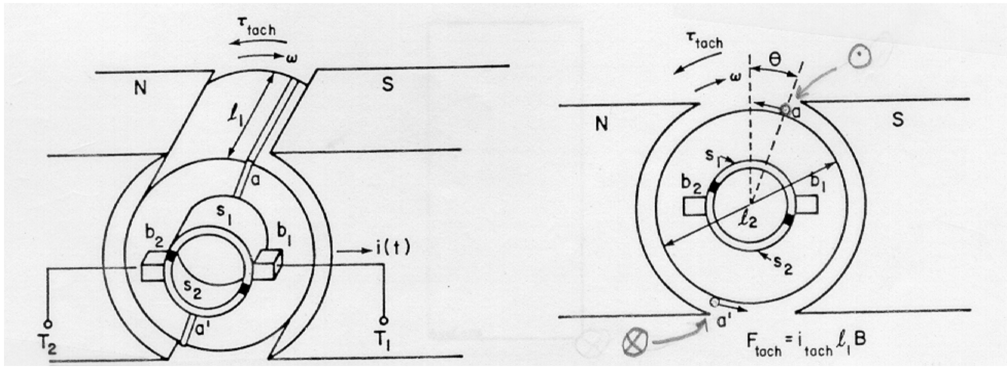


FIGURE 1.27. Cutaway view of the DC tachometer

By the sign convention for Faraday's law, the current in the tachometer is out of the page $\odot$ at the point a in Figure 1.27 and into the page $\otimes$ at a'. This current is opposite in direction to that of the current in the motor. The torque on the loop is $\tau_{tach} = -2(\ell_2/2)i_{tach}(t)\ell_1 B = -\ell_1\ell_2 B i_{tach}(t)$ which opposes the rotation of the shaft. If the terminals T_1, T_2 are connected to an amplifier with input impedance R_{in}, then an equivalent circuit is given in Figure 1.28.

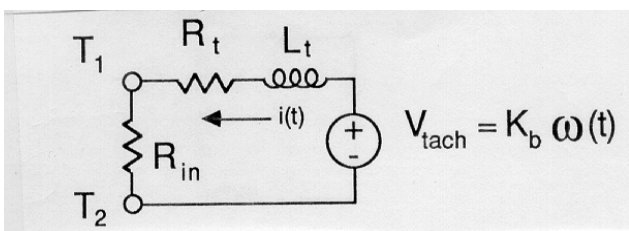


FIGURE 1.28. Equivalent Circuit

The mechanical power absorbed by the tachometer is equal to the electrical energy it produces as

$$\tau_{tach}(t)\omega_R(t) = -\ell_1\ell_2 B i_{tach}(t)\omega_R(t) = -V_{tach}(t)i_{tach}(t)$$

That is, the mechanical power done to oppose the torque τ_{tach} is converted to electrical power $V_{tach}(t)i_{tach}(t)$! Again, the tachometer has the same physical structure as the motor and is just acting as a generator putting out a voltage proportional to the angular speed. The only difference between a motor and a generator is how the device is used. In a motor, a voltage is applied to the terminals of the brushes to cause the armature (i.e., the current loops on the rotor) to turn. On the other hand, in a generator, the armature is (externally) turned to produce a voltage between the terminals of the brushes.

1.8 The Multi-Loop Motor*

The above single-loop motor was used to illustrate the basic Physics of the DC motor. However, it is not practical and first thing that must be done is to add more loops to extract more torque from the machine. Further, in the single-loop motor, the magnetic field due to the current in the *rotor* is an external magnetic field acting on the field windings. As the loop rotates, this magnetic field produces a changing flux in the field windings which in turn induces an emf in the field windings. This emf is referred to as the *armature reaction*. (The term armature refers to the rotating current loop, and reaction refers to the induced emf in the field windings due to the current in the rotor loop.) The armature reaction makes it difficult to maintain a constant field current. These problems can also be alleviated by adding more loops to the motor.

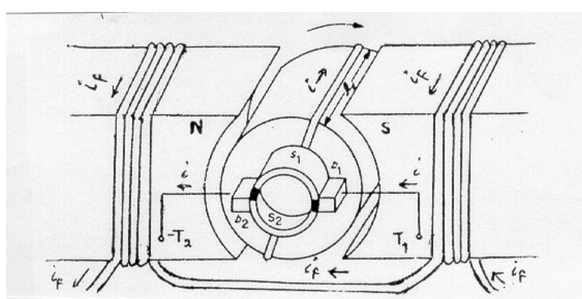


FIGURE 1.29. Single-loop motor with a field winding

1.8.1 *Increased Torque Production*

Figure 1.30 below shows the addition of several loops to the motor [15].

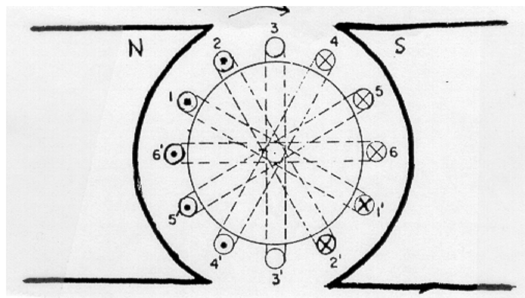


FIGURE 1.30. Redraw for 4 loops. Multi-Loop Armature

The torque on the rotor is now $\tau_m = n\ell_1\ell_2 Bi(t)$, where n is the number of rotor loops and B is the external magnetic field. Of course, some method will need to be given to make sure the current in each loop is reversed every half-turn. That is, all the loop sides under the south pole face must have their current into the page $\otimes$ and the loop sides under the north pole face must have their current out of the page $\odot$.

1.8.2 *Commutation of the Armature Current*

Each loop in the rotor must have the current in it switched every half-turn. This is done using a commutator which is illustrated in Figure 1.31 This is a commutator for the rotor shown in Figure 1.32 [5]. This rotor consists of 4 sets of rotor loops whose sides are 45° apart. As shown, the commutator for this rotor consists of 8 copper segments (labeled $a-h$ in 1.32a) which are separated by insulating material. Each commutator segment is connected to one of the ends of two rotor loops as shown in Figure 1.32a [5]. The commutator and rotor loops all rotate together rigidly while the two brushes (labeled b_1 and b_2) remain stationary. The two brushes are typically made of a carbon material and are mechanically pressed against the commutator surface making electrical contact[6].

[6] The figure shows a gap between the brushes and the commuator, but this was done for illustration and there is no gap in reality. Also, for illustrative purposes, the brushes are shown insider the commutator when in fact they are normally pressed against the commutator from the outside.

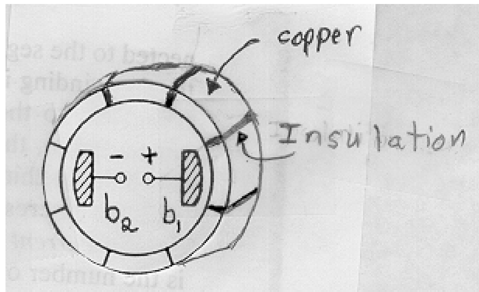


FIGURE 1.31. Commutator for the rotor in Figure 1.32

To get positive torque, it must be that whenever a side of the loop is under a south pole face, the current must be into the page ($\otimes$) and the other side of the loop (which is under the north pole face) must have its current out of the page ($\odot$). When the loop side passes through from being under one pole face to the other pole face, the current in that loop must be reversed or commutated. To understand how all this is done, consider the armature[7] in the position shown in Figure 1.32a. The current enters brush b_1 and into commutator segment c. By symmetry, half the current $i/2$ goes through loop $3-3'$ into commutator segment d then through loop $4-4'$ into commutator segment e, then through loop $5-5'$ into commutator segment f, then through loop $6-6'$ into commutator segment g and finally out through brush b_2. This path (circuit) of the current is denoted in **bold**. Similarly, there is a parallel path for the other half of the current armature current. Specifically, $i/2$ goes through loop $2'-2$ into commutator segment b then through loop $1'-1$ into commutator segment a then through loop $8'-8$ into commutator segment h then into loop $7'-7$ into commutator segment g and finally out through brush b_2. This path (circuit) is denoted in unbold. Consequently, this shows that for the rotor in the position shown in Figure 1.32a, there are two parallel paths each carry half the armature current through the loops. All loops that have a side under the south pole face have their current into the page and the other side of the loops (under the north pole face) have their current out of page so that positive torque is produced.

[7]The armature is the complete rotor assembly consisting of the commutator, rotor loops and rotor's magnetic core.

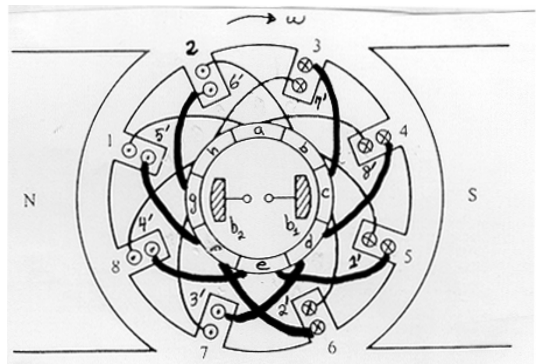


FIGURE 1.32. a Rotor loops and commutator for 4 sets of rotor loops. Brushes remained fixed in space (i.e., they do not rotate).

The sides of the loops in Figure 1.32a are 45° apart. Figure 1.32b below shows the rotor turned 45°/2 with respect to Figure 1.32a. In this case, brush b_1 shorts the two commutator segment b and c together while the brush b_2 shorts together the two commutator segments f and g. The ends of loop $2-2'$ are connected to commutator segments b and c (which are now shorted together) so that the current in this loop is now zero. Similarly, the ends of loop $6-6'$ are connected to commutator segments f and g and the current in this loop is also zero. For the remaining loops, $i/2$ goes through loop $3-3'$ into commutator segment d, then through loop $4-4'$ into commutator segment e, then into loop $5-5'$ into commutator segment f, and finally out brush b_2. These are denoted in **bold** in the figure. Similarly, $i/2$ goes through loop $1'-1$ into commutator segment a, then through loop $8'-8$ into commutator segment h, then into loop $7'-7$ into commutator segment g, and finally out brush b_2.

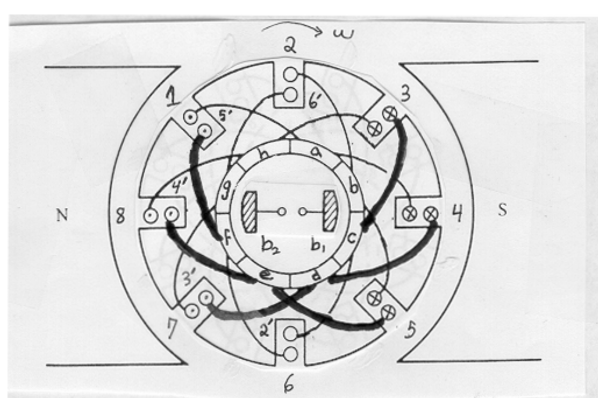


Figure 1.32b Rotor turned 45°/2 with respect to Figure 1.32a.

The motor continues to rotate and consider it now after it has moved additional $45°/2$ so that it has the position shown in Figure 1.32c below. In this case, the current enters brush b_1 and into commutator segment b. By symmetry, half the current $i/2$ goes through loop $2 - 2'$ into commutator segment c, then through loop $3 - 3'$ into commutator segment d, then through loop $4 - 4'$ into commutator segment e, then through loop $5 - 5'$ into commutator segment f, and finally out through brush b_2. This path (circuit) of the current is denoted in **bold**. Similarly, the other half of the current goes through loop $1' - 1$ into commutator segment a then through loop $8' - 8$ into commutator segment h then through loop $7' - 7$ into commutator segment g then into loop $6' - 6$ into commutator segment f and finally out through brush b_2. This path (circuit) is denoted in unbold.

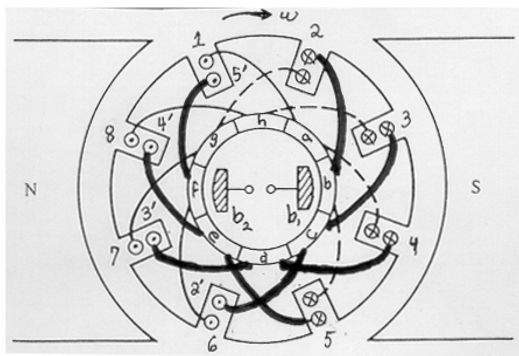


Figure 1.32c. Rotor turned 45° with respect to Figure 1.32a.

As this sequence of figures shows, the current in loops $2 - 2'$ and $6 - 6'$ were reversed as these two loops went through the vertical position. In general, there are two parallel paths each consisting of four loops and when any loop goes to the vertical position, the current in that loop is reversed. In this way, all sides of the loop under the south pole have keep their current into the page and all sides under the north pole have their current coming out of the page for positive torque production.

Remark The scheme for current commutation presented here is from [5]. However, there are many other schemes and the reader is referred to [5][7][8][10] for an introduction to the various schemes that are used. See especially [8] for a discussion of how commutation is often carried out in permanent magnet DC motors.

1.8.3 Armature Reaction

The left side of Figure 1.33 shows the magnetic field distribution in the iron and airgap due to the just the field current, while the right side of 1.33 shows the magnetic field distribution in the iron and airgap due to just the armature currents. As the right figure shows, the magnetic field distribution in the iron due to the armature current is perpendicular to the magnetic field distribution due to the field windings. As a result of this configuration, any changing magnetic field due to the armature current cannot induce any voltages in the field windings. Finally, Figure 1.34 shows the sum of the two magnetic fields. In summary, the symmetric placement of rotor loops around the periphery of the rotor essentially eliminates the armature reaction.

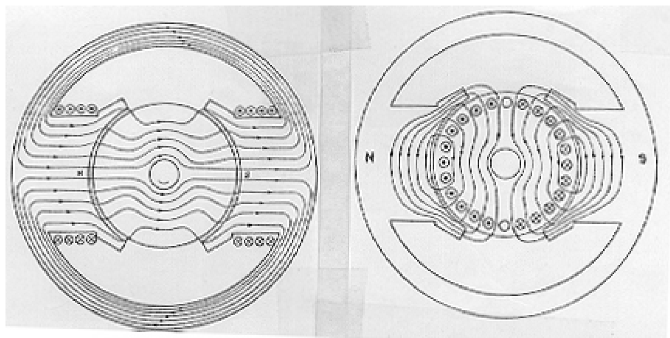


FIGURE 1.33. Redraw for 4 loops. Magnetic Field due to Field Current [16]

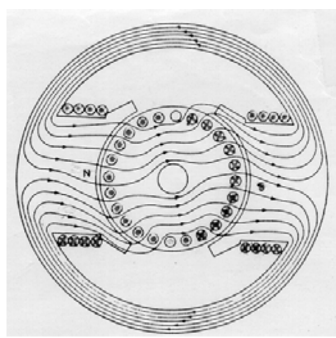


FIGURE 1.34. Redraw for 4 loops. Magnetic Field due to Field and Armature Currents [16]

1.8.4 Field Flux

The flux linkage in the field windings is

$$\varphi_f(i_f) = N_f \phi_f = N_f S B_f(i_f)$$

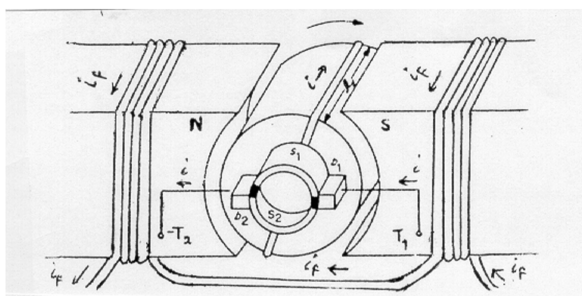
where N_f is the total number of field windings, S is the cross sectional area of the iron core of the field winding and $B_f(i_f)$ is the magnetic field produced *inside the magnetic-material* of the field circuit due to the field current i_f. The lines of the B field are shown in Figure 1.3, and, in this case, conservation of flux[8] implies the flux in each of the windings of the field is the same as the flux in the airgap under a north or south pole face. That is,

$$\phi_f = SB_f(i_f) = \pi(\ell_1/2)\ell_2 B(i_f)$$

were $\ell_1/2$ is the radius of the rotor, ℓ_2 is the axial length of the rotor and $B(i_f)$ is the radial magnetic field *in the airgap* due to the field current i_f. Consequently,

$$B(i_f) = \frac{SB_f(i_f)}{\pi\ell_1\ell_2/2} = \frac{\varphi_f(i_f)}{N_f\pi\ell_1\ell_2/2}$$

is an expression for the radial magnetic field *in the airgap* in terms of the flux linkage $\varphi_f(i_f) = N_f SB_f(i_f)$ in the field windings.



Seperately Excited DC Motor

1.8.5 Armature Flux

In what follows, the multiloop motor of Figure 1.32 is considered in which the armature circuit consists of two parallel circuits each having n loops. Recall that for the single-loop motor, the flux $\phi_{sl}(i_f, \theta_R)$ in the loop of the rotor due to the external magnetic field $B(i_f)$ is

$$\phi_{sl}(i_f, \theta_R) = -\ell_1\ell_2 B(i_f)(\theta_R \bmod \pi - \pi/2).$$

In the multi-loop motor, θ_R is hereby referenced relative to loop $1 - 1^{'}$ so that $\theta_R = 0$ corresponds to loop $1 - 1^{'}$ being vertical. Then, the k^{th} rotor loop is at (see Figure 1.32)

$$\theta_k = \theta_R + (k-1)\pi/n \text{ for } k = 1, ..., n$$

[8] This will be explained in Chapter 3.

where n is the number of loops in each parallel circuit[9]. The flux in the k^{th} rotor loop is

$$\begin{aligned}\phi_k(i_f,\theta_R) &= -\ell_1\ell_2 B(i_f)\left(\theta_k \bmod \pi - \frac{\pi}{2}\right)\\ &= -\ell_1\ell_2 B(i_f)\left(\left\{\theta_R + (k-1)\frac{\pi}{n}\right\} \bmod \pi - \frac{\pi}{2}\right).\end{aligned}$$

The total flux linkage $\phi(i_f,\theta_R)$ in the n rotor loops[10] is then

$$\begin{aligned}\phi(i_f,\theta_R) &= \sum_{k=1}^{n}\phi_k(i_f,\theta_R)\\ &= -\sum_{k=1}^{n}\ell_1\ell_2 B(i_f)\left(\left\{\theta_R + (k-1)\frac{\pi}{n}\right\} \bmod \pi - \frac{\pi}{2}\right). \quad (1.2)\end{aligned}$$

Recall that the sign convention for the fluxes $\phi_k(i_f,\theta_R)$ is such that if $-d\phi_k(i_f,\theta_R)/dt > 0$, then it is acting in the direction opposite to positive current flow, i.e., its sign convention is opposite to that of applied voltage v. In other words, the sum $v - (-d\phi(i_f,\theta_R)/dt) = v + d\phi(i_f,\theta_R)/dt$ is the total emf in the loop due to the applied voltage and the external magnetic field.

For $n = 4$, the normalized flux linkage $\phi(i_f,\theta_R)/(\ell_1\ell_2 B(i_f))$ is plotted as a function of the rotor position θ_R in Figure 1.35. Note that $\partial\phi(i_f,\theta_R)/\partial\theta_R = -n\ell_1\ell_2 B(i_f)$ which is proportional to the number of rotor loops n and to the strength of the external magnetic field strength $B(i_f)$ in the airgap.

---

[9] Note that $n = 4$ in Figure 1.32.

[10] In Figure 1.32(a)(b)(c), each of the two parallel sets of n rotor loops has the flux linkage $\phi(i_f,\theta_r)$ in it.

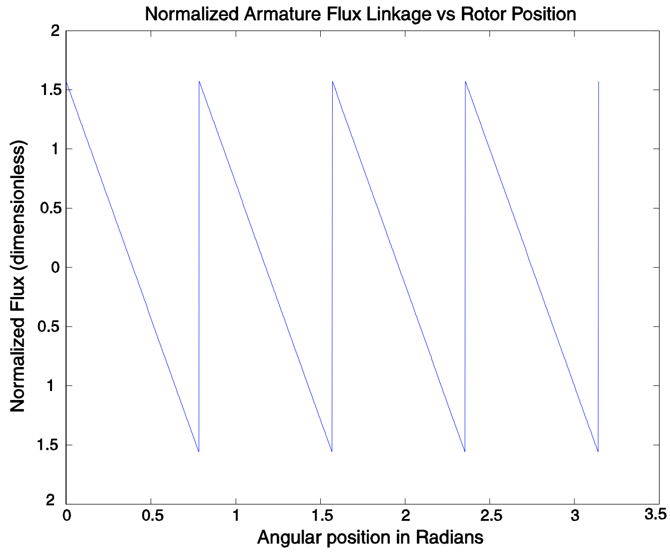


FIGURE 1.35. Normalized Flux Linkage $\phi(i_f, \theta_R)/(\ell_1 \ell_2 B(i_f))$ versus θ_R in radians with $n = 4$ sets of rotor loops and $0 \leq \theta_R \leq 2\pi$. The slope is $\partial\phi(i_f, \theta_R)/\partial\theta_R = -n\ell_1\ell_2 B(i_f)$.

1.8.6 *Dynamic Equations of the Separately Excited DC motor*

In the multi-loop motor considered here, the armature circuit consists of two parallel circuits each having n loops. That is, there are a total of $2n$ loops on the rotor as each of the parallel circuits has a loop at the same location on the rotor. Let L denote self inductance of the n rotor loops making up either of the two parallel circuits. Note that if a current $i/2$ is in each parallel circuit, then each circuit will have a flux linkage of $Li/2$ due to its current and an *additional* flux linkage of $Li/2$ due to the current in the other circuit. This is simply because the two sets of parallel loops (windings) are wound together so that they are perfectly (magnetically) coupled.

The total flux linkage in the n loops making up either of the parallel circuits is now computed. To proceed, let i be the current into the armature so that $i/2$ is the current in the n loops of each parallel circuit. The quantity $Li/2$ is the flux linkage due to the current $i/2$ in the loops and an additional flux linkage of $Li/2$ is produced in these same loops by the current $i/2$ in the *other* parallel circuit for a total flux of Li. Also $\phi(i_f, \theta_R)$ is the flux linkage in the n loops due to the *external* magnetic field, but recall that its sign convention for the induced emf is opposite to that of Li. Recall in section 1.3.3 that normal to the flux surface was taken to be radially in to compute the flux Li while in section 1.3.2 the surface normal was taken to be radially out to compute the flux $\phi(i_f, \theta_R)$. Simply writing $-\phi(i_f, \theta_R)$ then gives the flux due to the external magnetic field with the surface normal radially in. Now the induced voltages by both of these changing fluxes will have the same sign convention as the applied armature voltage. The total flux linkage is either parallel circuit is then written as

$$Li - \phi(i_f, \theta_R).$$

Let R_1 denote the resistance of the n loops connected in series making up each parallel circuit. The equation describing the electrical dynamics of the armature current in each of the parallel circuits is found by applying Kirchoff's voltage law to get

$$-\frac{d}{dt}\Big(Li - \phi(i_f, \theta_R)\Big) - R_1 i/2 + v_a = 0$$

where v_a is the applied voltage to the armature. Finally, defining $R \triangleq R_1/2$, the equation describing the electrical dynamics of the armature circuit is

$$L\frac{di}{dt} = -Ri + \frac{d\phi(i_f, \theta_R)}{dt} + v_a. \tag{1.3}$$

The quantity $d\phi(i_f, \theta_R)/dt$ can be expanded to obtain

$$\frac{d\phi(i_f, \theta_R)}{dt} = \frac{\partial \phi}{\partial i_f}\frac{di_f}{dt} + \frac{\partial \phi}{\partial \theta_R}\omega_R = \frac{\partial \phi}{\partial i_f}\frac{di_f}{dt} - n\ell_1\ell_2 B(i_f)\omega_R. \tag{1.4}$$

Here $B(i_f)$ is the strength of the radial magnetic field *in the airgap* produced by the current in the field windings. The flux linkage in the field windings is $\varphi_f(i_f) = N_f S B_f(i_f)$ where $B_f(i_f)$ is the magnetic field strength *in the iron core* of the field and S is the cross sectional area of this iron core. The flux $SB_f(i_f)$ goes through the airgap and by conservation of flux (see section 1.8.4), $SB_f(i_f) = (\pi\ell_1\ell_2/2)\,B(i_f)$ so that $B(i_f) = \frac{N_f}{N_f}\frac{SB_f(i_f)}{\pi\ell_1\ell_2/2} = \frac{\varphi_f(i_f)}{N_f\pi\ell_1\ell_2/2}$. Consequently,

$$n\ell_1\ell_2 B(i_f) = n\ell_1\ell_2\frac{\varphi_f(i_f)}{N_f\pi\ell_1\ell_2/2} = K_m\varphi_f(i_f),\ K_m \triangleq \frac{2}{\pi}\frac{n}{N_f}.$$

As a result, equation (1.4) may be rewritten as

$$\frac{d\phi(i_f, \theta_R)}{dt} = \frac{\partial \phi}{\partial i_f}\frac{di_f}{dt} - K_m\varphi_f(i_f)\omega_R.$$

Each loop carries the current $i/2$ so that the torque produced by the two sides of each loop is $2\,(\ell_2/2)\,(i/2)\ell_1 B(i_f)$. As there are $2n$ loops, the total torque is

$$\tau_m = 2n\ell_1\ell_2 B(i_f)(i/2) = n\ell_1\ell_2 B(i_f)i = K_m\varphi_f(i_f)i \tag{1.5}$$

using the above expressions for K_m and $\varphi_f(i_f)$. Consequently, using (1.3)(1.5), the complete set of equations for the separately excited DC motor is then

$$L\frac{di}{dt} = -Ri + \frac{\partial\phi}{\partial i_f}\frac{di_f}{dt} - K_m\varphi_f(i_f)\omega_R + v_a \tag{1.6}$$

$$J\frac{d\omega_r}{dt} = K_m\varphi_f(i_f)i - \tau_L \tag{1.7}$$

$$\frac{d\varphi_f(i_f)}{dt} = -R_f i_f + v_f \tag{1.8}$$

where v_f is the applied voltage to the field, R_f is the resistance of the field winding, τ_L is the load torque on the motor and J is the rotor's moment of inertia.

If the iron in the field is not in saturation, then one may write

$$\varphi_f(i_f) = L_f i_f$$

and the dynamic model simplifies to

$$L\frac{di}{dt} = -Ri + \frac{\partial\phi}{\partial i_f}\frac{di_f}{dt} - K_m L_f i_f\omega_R + v_a \tag{1.9}$$

$$J\frac{d\omega_r}{dt} = K_m L_f i_f i - \tau_L \tag{1.10}$$

$$L_f\frac{di_f}{dt} = -R_f i_f + v_f \tag{1.11}$$

Under normal operating conditions, the term $\frac{\partial\phi}{\partial i_f}\frac{di_f}{dt}$ is typically negligible and so the model reduces to[11]

$$L\frac{di}{dt} = -Ri - K_m L_f i_f\omega_R + v_a \tag{1.12}$$

$$J\frac{d\omega_r}{dt} = K_m L_f i_f i - \tau_L \tag{1.13}$$

$$L_f\frac{di_f}{dt} = -R_f i_f + v_f \tag{1.14}$$

A typical mode of operation is to use the field voltage v_f to hold the field current i_f constant at some value. Then the field flux is of course constant and $L_f di_f/dt = 0$. However, in the first equation (1.6) (or 1.12), it is seen that the back-emf (voltage) is $-K_m\varphi_f\omega_R$ (or $-K_m L_f i_f\omega_R$) which increases in proportion to the speed. The input voltage v_a must be at least large enough to overcome the back-emf in order to maintain the armature current. To have the motor achieve higher speeds within the voltage limit

[11]It will be seen later that field oriented control of an induction motor results in a mathematical model of the induction motor that looks similar to these equations!

$|v_a| \leq V_{\max}$, *field weakening* is employed. This is accomplished by decreasing the field flux $\varphi_f = L_f i_f$ at higher speeds usually according to the following flux reference

$$\begin{aligned}\varphi_{ref} &= \varphi_{f0} \qquad \text{for } |\omega_R| \leq \omega_{base} \\ &= \varphi_{f0}\frac{\omega_{base}}{\omega_R} \quad \text{for } |\omega_R| \geq \omega_{base}.\end{aligned} \tag{1.15}$$

Field weakening results in the back-emf $-K_m\varphi_f\omega = K_m\varphi_{f0}\frac{\omega_{base}}{\omega_R}\omega_R = -K_m\varphi_{f0}\omega_{base}$ being constant for speeds greater than ω_{base}. The trade-off is that the torque $K_m\varphi_f i_a$ is less for the same i_a due to the decrease in field flux linkage φ_f. If the armature resistance is negligible and i_f is constant, then the *base speed* ω_{base} is defined to be the speed satisfying $K_m\varphi_f\omega_{base} = V_{\max}$. Otherwise, the base speed is chosen to be somewhat smaller to account for the Ri and $\frac{\partial\phi}{\partial i_f}\frac{di_f}{dt}$ drop (see equation (1.6)).

Permanent Magnet DC Motor

The equations describing a multi-loop *permanent magnet* DC motor are easily derived from the above derivation of the equations for the separately excited DC motor. In this case, the electromagnet (iron plus field winding) is replaced by a permanent magnet so that $B(i_f)$ is simply replaced by the constant value B of the permanent magnet's field strength in the airgap. Thus (1.2) becomes

$$\phi(\theta_R) = -\sum_{k=1}^{n}\ell_1\ell_2 B\left(\left\{\theta_R + (k-1)\frac{\pi}{n}\right\}\bmod \pi - \frac{\pi}{2}\right).$$

Kirchoff's voltage law applied to the armature circuit gives[12]

$$-Ldi/dt - (-d\phi(\theta_R)/dt) - Ri + v_a = 0.$$

The equations for a multi-loop $PM\ DC$ motor are then

$$\begin{aligned}\frac{d}{dt}Li - \frac{d}{dt}\phi(\theta_R) &= -Ri + v_a \\ J\frac{d\omega_R}{dt} &= n\ell_1\ell_2 Bi - \tau_L\end{aligned}$$

or, as $\partial\phi(\theta_R)/\partial\theta_R = -n\ell_1\ell_2 B$, this becomes

$$L\frac{di}{dt} = -Ri - K_b\omega_R + v_a \tag{1.16}$$

$$J\frac{d\omega}{dt} = K_T i - \tau_L \tag{1.17}$$

[12] Recall that the sign convention for $-d\phi(\theta_R)/dt$ is opposite to that of $-Ldi/dt$ and hence the sum of these two emfs is $-Ldi/dt - (-d\phi(\theta_R)/dt)$.

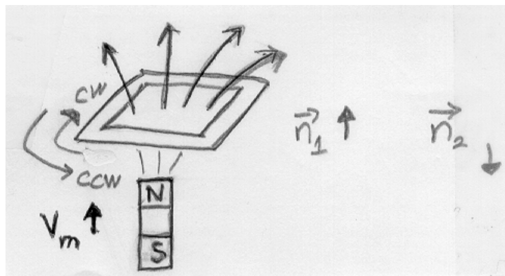
where $K_b = |\partial\phi(\theta_R)/\partial\theta_R| = n\ell_1\ell_2 B = K_T$. This system of equations is of the same form as derived previously for the single-loop motor.

1.9 Problems

Faraday's Law and Induced Electromotive Force (emf)

Problem 1 Faraday's Law

Consider the figure below where the magnet is moving up *into the square planar loop of copper wire.*



Using the normal $\vec{\mathbf{n}}_1$, is the flux in the loop produced by the magnet increasing or decreasing?

Using the normal $\vec{\mathbf{n}}_1$, what is the direction of positive travel around the surface whose boundary is the loop? (clockwise or counterclockwise)

What is the direction of the induced current on the figure.(clockwise or counterclockwise). Does the induced current produce a flux in the loop that opposes the flux produced by the magnet?

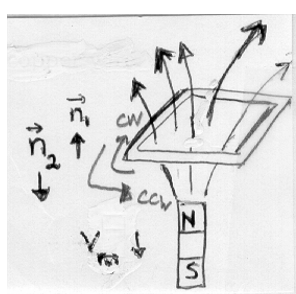
Using the normal $\vec{\mathbf{n}}_2$, is the flux increasing or decreasing?

Using the normal $\vec{\mathbf{n}}_2$, what is the direction of positive travel around the surface whose boundary is the loop? (clockwise or counterclockwise)

What is the direction of the induced current on the figure.(clockwise or counterclockwise). Does the induced current produce a flux in the loop that opposes the flux produced by the magnet?

Problem 2 Faraday's Law

Consider the figure below where the magnet is moving down *away from the loop of copper wire.*



Using the normal $\vec{\mathbf{n}}_1$*, is the flux in the loop produced by the magnet increasing or decreasing?*

Using the normal $\vec{\mathbf{n}}_1$*, what is the direction of positive travel around the surface whose boundary is the loop? (clockwise or counterclockwise)*

What is the direction of the induced current on the figure.(clockwise or counterclockwise). Does the induced current produce a flux in the loop that opposes the flux produced by the magnet?

Using the normal $\vec{\mathbf{n}}_2$*, is the flux increasing or decreasing?*

Using the normal $\vec{\mathbf{n}}_2$*, what is the direction of positive travel around the surface whose boundary is the loop? (clockwise or counterclockwise)*

What is the direction of the induced current on the figure.(clockwise or counterclockwise). Does the induced current produce a flux in the loop that opposes the flux produced by the magnet?

Problem 3 The Linear DC Motor

Consider the simple linear DC motor analyzed in the text.

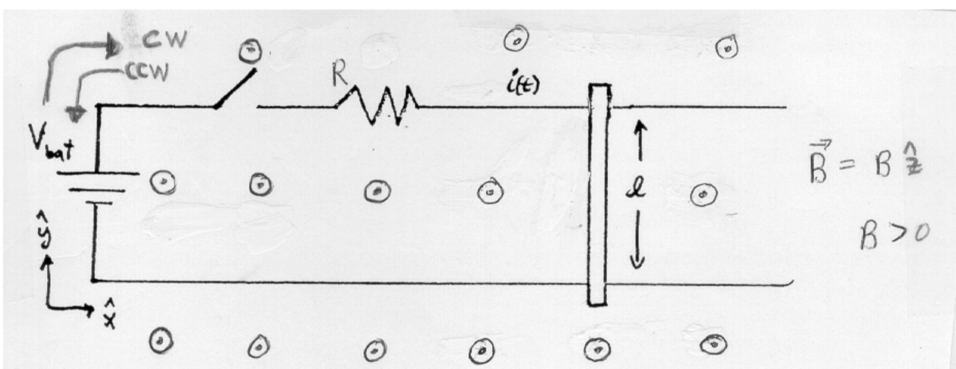
With the normal to the surface enclosed by the loop taken to be $\vec{\mathbf{n}} = -\hat{\mathbf{z}}$*, what is the flux through the surface?*

What it the direction of positive travel around this flux surface?

What is the induced emf ξ *in the loop in terms of* B*,* ℓ *and the speed* v *of the bar?*

Do V_{bat} *(the external voltage applied to the loop) and* ξ *have the same sign convention? Explain why* ξ *is now negative.*

Problem 4 The Linear DC Motor



Consider the simple linear motor in the figure where the magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{z}}$ $(B > 0)$ *is up out of the page.*

Closing the switch causes a current to flow in the wire loop. What is the magnetic force $\vec{\mathbf{F}}_{magnetic}$ *on the sliding bar in terms of* B*,* i*, and* ℓ *(the length of the bar)? Give both the magnitude and direction of* $\vec{\mathbf{F}}_{magnetic}$*.*

With the normal to the surface enclosed by the loop taken to be $\vec{\mathbf{n}} = \hat{\mathbf{z}}$*, what is the flux through the surface?*

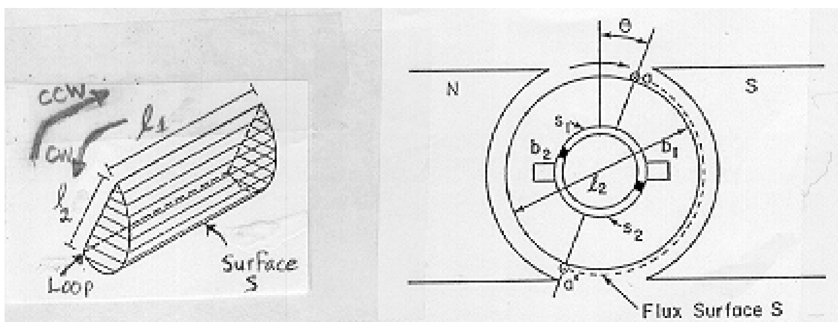
What is the induced emf ξ in the loop in terms of B, ℓ and the speed v of the bar?

What is the sign convention for the induced emf ξ drop around the loop? (That is, if $\xi > 0$, would it act to push current in the clockwise or counter-clockwise direction?).

Do V_{bat} (the external voltage applied to the loop) and ξ have the same sign convention?

Problem 5 Back-Emf in the Single-Loop Motor

Consider the single-loop motor with the flux surface as indicated. A voltage source connected to the brushes is forcing current down side a (indicated with a $\otimes$) and up side a' (indicated with a $\odot$).

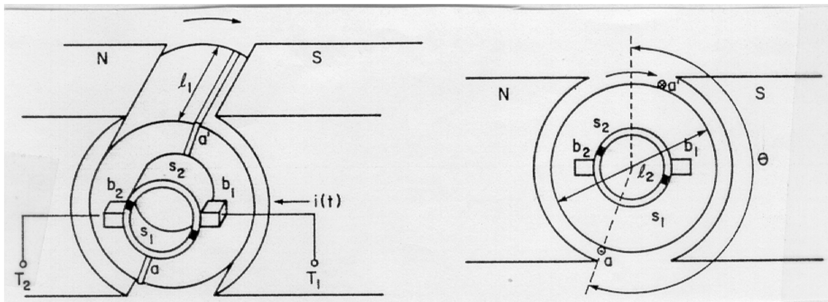


With the motor at the angular position θ_R shown and using the ***inward*** *normal (i.e., $\vec{\mathbf{n}} = -\hat{\mathbf{r}}$), compute the flux through the surface in terms of the magnitude B of the radial magnetic field in the air gap, the axial length ℓ_1 of the motor, the diameter ℓ_2 of the motor and the angle θ_R of the rotor.*

What is the positive direction of travel around the flux surface $\mathcal{S}$? (direction 1 or direction 2)

What is the emf ξ induced in the rotor loop? What is the sign convention for the induced emf ξ drop around the loop? (That is, if $\xi > 0$, would it act to push current in CW direction or the CCW direction?). Do v_a (the external voltage applied to the loop) and ξ have the same sign convention? Explain why ξ is now negative.

Problem 6 *The figure below shows the rotor loop where $\pi < \theta_R < 2\pi$.*



DRAW FLUX SURFACE Rotor loop where $\pi < \theta_R < 2\pi$

Using the flux surface show in the figure, show that $\phi(\theta_R) = -\ell_1\ell_2 B\left(\theta_R - \pi - \frac{\pi}{2}\right)$ *for* $\pi < \theta_R < 2\pi$.

Combine this with the result in the text to show that $\phi(\theta_R) = -\ell_1\ell_2 B\left(\theta_R \bmod \pi - \frac{\pi}{2}\right)$ *for all* θ_R.

Plot $\phi(\theta_R)/(\ell_1\ell_2 B)$ *for* $0 \le \theta_R \le 2\pi$.

Multi-loop Motor

Problem 7 Neutral Plane and Brush Shifting

In the commutation scheme for the multi-loop motor, it was shown that when a rotor loop was perpendicular to the brushes, the current in the loop was shorted out (brought to zero). Consequently, it is highly desirable that the total induced voltage in that loop be as close to zero as possible to prevent arcing. Figure 1.34 shows the total *B field distribution in the DC machine. If the armature current were zero, then the field would be horizontal as in Figure 1.33a. At very high armature currents (e.g., in large machines used in heavy industry), the field is skewed as shown in Figure 1.34. The neutral plane is the plane cutting through the axis of the rotor for which the* total *B field is perpendicular to the plane. The* k^{th} *rotor loop*[13] *has the flux* $L_1 i - \phi_k(i_f, \theta_R)$ *in it (*L_1 *is the inductance of an individual loop) so that the induced voltage in this loop is* $-d(L_1 i - \phi_k(i_f, \theta_R))/dt$. *In the following, assume that the field current* i_f *is kept constant.*

Explain why $L_1 i - \phi_k(i_f, \theta_R)$ *is a maximum (or minimum) as a function of the rotor position* θ_R *when the* k^{th} *loop coincides with the neutral plane.*

Explain why $-\partial(L_1 i - \phi_k(i_f, \theta_R))/\partial\theta_R = 0$ *when the* k^{th} *loop coincides with the neutral plane.*

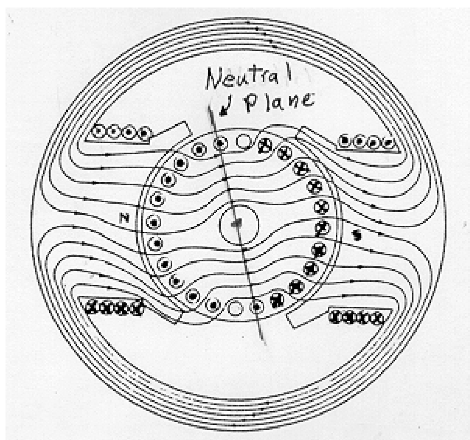
Shift (rotate) the brushes so that the neutral plane is perpendicular to the brushes, that is, in the figure below, the brushes are rotated counterclockwise so that the plane *of the loop undergoing commutation is normal to the* total *B field. Show then that the induced voltage in the* k^{th} *loop when it undergoes*

[13] The flux $\phi_k(i_f, \theta_R)$ has been multiplied by -1 because then $-\phi_k(i_f, \theta_R)$ and $L_1 i$ have the same reference directions for the induced voltages they produce.

commutation is

$$-\frac{d(L_1 i - \phi_k(i_f, \theta_R))}{dt} = -L_1 \frac{di}{dt}$$

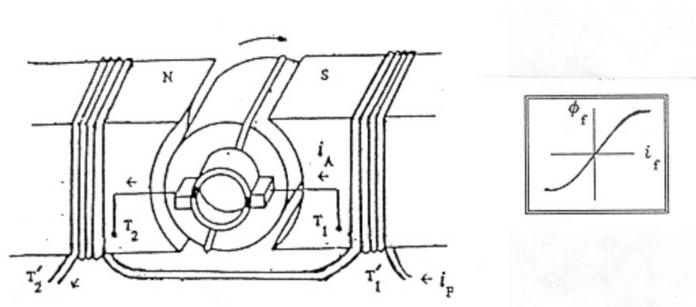
Explain why the shifting of the brushes is a good idea to alleviate arcing.
Eoes the amount that the brushes are to be rotated depend on the amount of armature current?



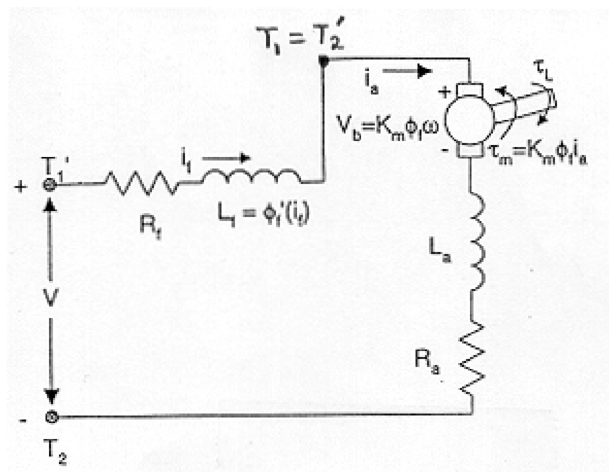
Redraw with only 8 *loops. Magnetic Field due to Field and Armature Currents*

Problem 8 Conservation of Energy in the Separately Excited DC Motor
Using the equations (1.6)(1.7)(1.8) of the separately excited DC motor show that energy conservations holds. Give a physical interpretation to the various expressions.

Problem 9 Series DC motor [1]
In a separately excited DC motor, connect the terminal T_1 *of the armature to the terminal* T_2' *of the field circuit and apply a single voltage source between the remaining terminals* T_1', T_2. *This configuration is referred to as a* series DC motor *and has traditionally been used in traction drives. An equivalent circuit is also shown.*
Let $\varphi(i_f) = L_f i_f$ *and derive the equations that characterize this system.*
Show that the torque cannot change sign (i.e., it is always positive or always negative).
What must be done to change the sign of the torque?



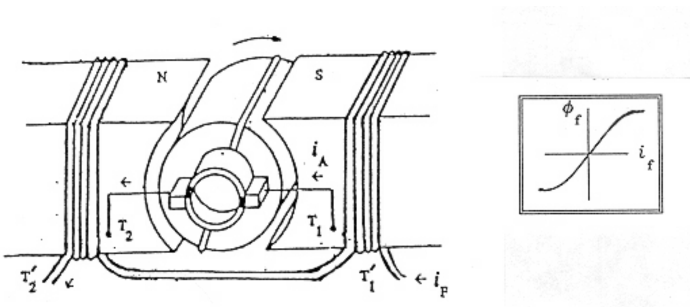
Seperately Excited DC Motor



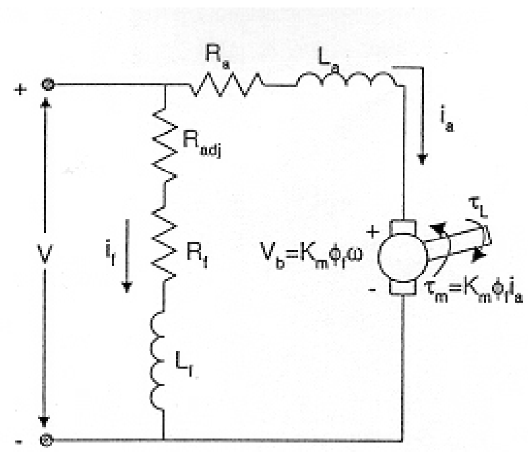
Schematic for a Series DC motor

Problem 10 Shunt DC motor [2]

In a separately excited DC motor, connect the terminal T_1 *of the armature to the terminal* T_1' *of the field circuit and similarly connect the other two terminals* T_2, T_2' *together. This configuration is referred to as a* shunt DC motor *for which an equivalent circuit is shown below. The resistance* R_{adj} *is an adjustable resistance added in series with the field winding and is used to make the control of the motor easier.*



Seperately Excited DC Motor



Schematic for a Shunt DC Motor

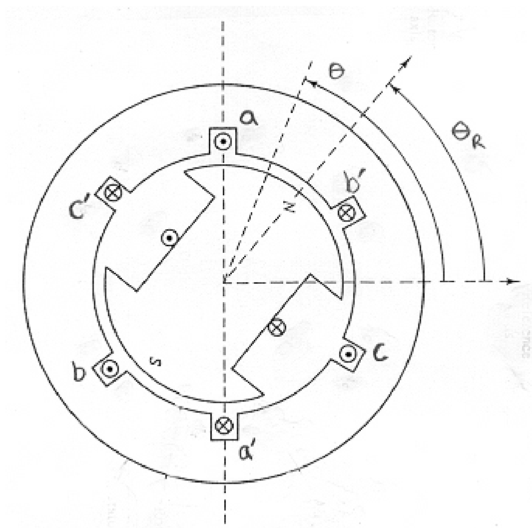
Let $\varphi(i_f) = L_f i_f$ and derive the equations that characterize this system.

Problem 11 A 3-phase generator [17]

Consider a simplified model of a 3-phase generator shown in the figure below. The current i_f around the rotor iron is held constant. With θ, θ_R defined as in the figure, it turns out that the magnetic field in the air gap due to i_f is given by

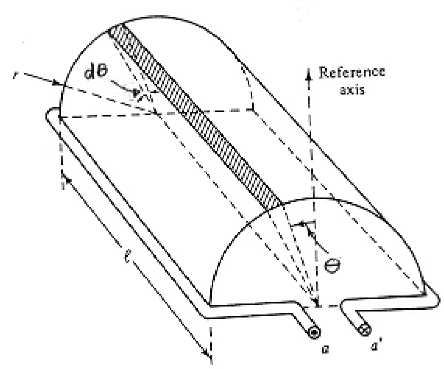
$$\vec{\mathbf{B}}(\theta) = B_{R\max}\cos(\theta - \theta_R)\hat{\mathbf{r}}$$

where $B_{R\max} > 0$ is constant when i_f is constant.



A simple 3-phase generator.

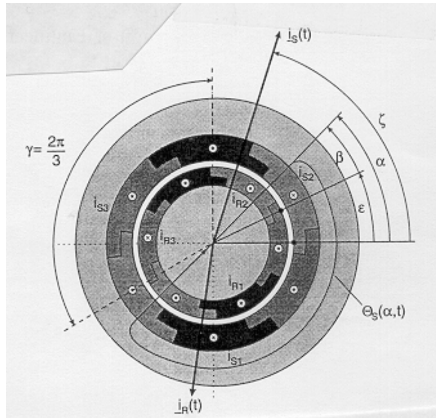
Using the flux surface shown below, compute the flux in the stator loop $a - a\prime$ *and the voltage induced in this loop.*



Also compute the fluxes and voltages in phases $b - b\prime$ *and* $c - c\prime$*. With the ends* $a\prime, b\prime, c\prime$ *tied together, show that if the rotor is going at constant speed, this results in a three phase generator in which the three voltages are identical except* $120°$ *out of phase with each other.*

Problem 12 *Three phase generator with distributed windings*

Consider the three phase generator with distributed windings as illustrated in the Figure below.



Need a PM rotor

With the rotor going at constant angular speed ω_R, find the three steady-state voltages $\xi_{1-1'}, \xi_{2-2'}, \xi_{3-3'}$ which are generated by this machine.